

# Computational Logic

## CLP Semantics and Fundamental Results

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### Constraint Domains

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- Semantics parameterized by the constraint domain:  
 $\text{CLP}(\mathcal{X})$ , where  $\mathcal{X} \equiv (\Sigma, \mathcal{D}, \mathcal{L}, \mathcal{T})$
- Signature  $\Sigma$ : set of predicate and function symbols, together with their arity
- $\mathcal{L} \subseteq \Sigma$ -formulae: constraints
- $\mathcal{D}$  is the set of actual elements in the domain
- $\Sigma$ -structure  $\mathcal{D}$ : gives the meaning of predicate and function symbols (and hence, constraints).
- $\mathcal{T}$  a first-order theory (axiomatizes some properties of  $\mathcal{D}$ )
- $(\mathcal{D}, \mathcal{L})$  is a *constraint domain*
- Assumptions:
  - ◇  $\mathcal{L}$  built upon a first-order language
  - ◇  $= \in \Sigma$  is identity in  $\mathcal{D}$
  - ◇ There are identically false and identically true constraints in  $\mathcal{L}$
  - ◇  $\mathcal{L}$  is closed w.r.t. renaming, conjunction and existential quantification

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## Domains (I)

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- $\Sigma = \{0, 1, +, *, =, <, \leq\}$ ,  $D = \mathbf{R}$ ,  $\mathcal{D}$  interprets  $\Sigma$  as usual,  $\mathfrak{R} = (D, \mathcal{L})$ 
  - ◇ Arithmetic over the reals
  - ◇ Eg.:  $x^2 + 2xy < \frac{y}{x} \wedge x > 0$  ( $\equiv xxx + xxy + xxy < y \wedge 0 < x$ )
- Question: is 0 needed? How can it be represented?

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- Let us assume  $\Sigma' = \{0, 1, +, =, <, \leq\}$ ,  $\mathfrak{R}_{Lin} = (D', \mathcal{L}')$ 
  - ◇ Linear arithmetic
  - ◇ Eg.:  $3x - y < 3$  ( $\equiv x + x + x < 1 + 1 + 1 + y$ )

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- Let us assume  $\Sigma'' = \{0, 1, +, =\}$ ,  $\mathfrak{R}_{LinEq} = (D'', \mathcal{L}'')$ 
  - ◇ Linear equations
  - ◇ Eg.:  $3x + y = 5 \wedge y = 2x$

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## Domains (II)

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- $\Sigma = \{<constant\ and\ function\ symbols>, =\}$
- $D = \{ \text{finite trees} \}$
- $\mathcal{D}$  interprets  $\Sigma$  as tree constructors
- Each  $f \in \Sigma$  with arity  $n$  maps  $n$  trees to a tree with root labeled  $f$  and whose subtrees are the arguments of the mapping
- Constraints: syntactic tree equality
- $\mathcal{FT} = (D, \mathcal{L})$ 
  - ◇ Constraints over the Herbrand domain
  - ◇ Eg.:  $g(h(Z), Y) = g(Y, h(a))$
- LP  $\equiv$  CLP( $\mathcal{FT}$ )

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## Domains (III)

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- $\Sigma = \{ \langle \text{constants} \rangle, \lambda, ., ::, = \}$
  - $D = \{ \text{finite strings of constants} \}$
  - $\mathcal{D}$  interprets  $.$  as string concatenation,  $::$  as string length
    - ◇ Equations over strings of constants
    - ◇ Eg.:  $X.A.X = X.A$
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- $\Sigma = \{0, 1, \neg, \wedge, =\}$
  - $D = \{ \text{true}, \text{false} \}$
  - $\mathcal{D}$  interprets symbols in  $\Sigma$  as boolean functions
  - $\text{BOOL} = (D, \mathcal{L})$ 
    - ◇ Boolean constraints
    - ◇ Eg.:  $\neg(x \wedge y) = 1$

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## CLP( $\mathcal{X}$ ) Programs

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- Recall that:
  - ◇  $\Sigma$  is a set of predicate and function symbols
  - ◇  $\mathcal{L} \subseteq \Sigma$ -formulae are the constraints
- $\Pi$ : set of predicate symbols definable by a program
- Atom:  $p(t_1, t_2, \dots, t_n)$ , where  $t_1, t_2, \dots, t_n$  are terms and  $p \in \Pi$
- Primitive constraint:  $p(t_1, t_2, \dots, t_n)$ , where  $t_1, t_2, \dots, t_n$  are terms and  $p \in \Sigma$  is a predicate symbol
- Every constraint is a (first-order) formula built from primitive constraints
- The class of constraints will vary (generally only a subset of formulas are considered constraints)
- A CLP program is a collection of rules of the form  $a \leftarrow b_1, \dots, b_n$  where  $a$  is an atom and the  $b_i$ 's are atoms or constraints
- A fact is a rule  $a \leftarrow c$  where  $c$  is a constraint
- A goal (or query)  $G$  is a conjunction of constraints and atoms

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## Basic Operations on Constraints

- Constraint domains are expected to support some basic operations on constraints
  1. Consistency (or satisfiability) test:  $\mathcal{D} \models \exists c$ ,
  2. Implication or entailment:  $\mathcal{D} \models c_0 \rightarrow c_1$ ,
  3. Projection of a constraint  $c_0$  onto variables  $\tilde{x}$  to obtain a constraint  $c_1$  such that  $\mathcal{D} \models c_1 \leftrightarrow \exists_{-\tilde{x}} c_0$ ,
  4. Detection of uniqueness of variable value:  $\mathcal{D} \models c(x, \tilde{z}) \wedge c(y, \tilde{w}) \rightarrow x = y$
- Actually, only the first one is really required
- In actual implementations, some of these operations—in particular the test of consistency—may be incomplete
- Examples:
  - ◊  $x * x < 0$  is inconsistent in  $\mathfrak{R}$  (because  $\neg \exists x \in \mathfrak{R} : x * x < 0$ )
  - ◊  $\mathcal{D} \models (x \wedge y = 1) \rightarrow (x \vee y = 1)$  in  $\mathcal{BOOL}$
  - ◊ In  $\mathcal{FT}$ , the projection of  $x = f(y) \wedge y = f(z)$  on  $\{x, z\}$  is  $x = f(f(z))$
  - ◊ In  $\mathcal{WE}$ ,  $\mathcal{D} \models x.a.x = x.a \wedge y.b.y = y.b \rightarrow x = y$
- Prove the last assertion!

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## Properties of CLP Languages

- $\mathcal{T}$  axiomatizes some of the properties of  $\mathcal{D}$
- For a given  $\Sigma$ , let  $(\mathcal{D}, \mathcal{L})$  be a constraint domain with signature  $\Sigma$ , and  $\mathcal{T}$  a  $\Sigma$ -theory.
- $\mathcal{D}$  and  $\mathcal{T}$  correspond on  $\mathcal{L}$  if:
  - ◊  $\mathcal{D}$  is a model of  $\mathcal{T}$ , and
  - ◊ for every constraint  $c \in \mathcal{L}$ ,  $\mathcal{D} \models \exists c$  iff  $\mathcal{T} \models \exists c$ .
- $\mathcal{T}$  is *satisfaction complete* with respect to  $\mathcal{L}$  if for every constraint  $c \in \mathcal{L}$ , either  $\mathcal{T} \models \exists c$  or  $\mathcal{T} \models \neg \exists c$ .
- $(\mathcal{D}, \mathcal{L})$  is *solution compact* if

$$\forall c \exists \{c_i\}_{i \in I} : \mathcal{D} \models \forall \tilde{x} \neg c(\tilde{x}) \iff \bigvee_{i \in I} c_i(\tilde{x})$$

i.e., any negated constraint in  $\mathcal{L}$  can be expressed as a (in)finite disjunction of constraints

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## Solution Compactness

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- Important to lift SLDNF results to  $\text{CLP}(\mathcal{X})$
- We have to deal only with user predicates
- E.g.
  - ◊  $x \not\leq y$  in  $\text{CLP}(\mathbb{R})$  is  $x < y$
  - ◊  $x \neq y$  in  $\text{CLP}(\mathbb{R})$  is  $x < y \vee y < x$
  - ◊  $\mathbb{R}_{Lin}$  with constraint  $x \neq \pi$  is not s.c.
- How can we express  $x \neq y$  in  $\text{CLP}(\mathcal{FT})$ ?

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## Logical Semantics (I)

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- Two common logical semantics exist.
- The first one interprets a rule

$$p(\tilde{x}) \leftarrow b_1, \dots, b_n$$

as the logic formula

$$\forall \tilde{x}, \tilde{y} \ p(\tilde{x}) \vee \neg b_1 \vee \dots \vee \neg b_n$$

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## Logical Semantics (II)

- The second one associates a logic formula to each predicate in  $\Pi$ 
  - ◊ If the set of rules of  $P$  with  $p$  in the head is:

$$\begin{aligned} p(\tilde{x}) &\leftarrow B_1 \\ p(\tilde{x}) &\leftarrow B_2 \\ &\vdots \\ p(\tilde{x}) &\leftarrow B_n \end{aligned}$$

then the formula associated with  $p$  is:

$$\begin{aligned} \forall \tilde{x} p(\tilde{x}) \leftrightarrow & \exists \tilde{y}_1 B_1 \\ & \vee \exists \tilde{y}_2 B_2 \\ & \vdots \\ & \vee \exists \tilde{y}_n B_n \end{aligned}$$

- ◊ If  $p$  does not occur in the head of a rule of  $P$ , the formula is:  $\forall \tilde{x} \neg p(\tilde{x})$
- ◊ The collection of all such formulas is the *Clark completion* of  $P$  (denoted by  $P^*$ )
- These two semantics differ on the treatment of the negation

## Logical Semantics (III)

- A *valuation* is a mapping from variables to  $D$ , and the natural extension which maps terms to  $D$  and formulas to closed  $\mathcal{L}^*$ -formulas.
- A  $\mathcal{D}$ -interpretation of a formula is an interpretation of the formula with the same domain as  $\mathcal{D}$  and the same interpretation for the symbols in  $\Sigma$  as  $\mathcal{D}$ .
- It can be represented as a subset of  $B_{\mathcal{D}}$  where

$$B_{\mathcal{D}} = \{p(\tilde{d}) \mid p \in \Pi, \tilde{d} \in D^k\}$$

- A  $\mathcal{D}$ -model of a closed formula is a  $\mathcal{D}$ -interpretation which is a model of the formula.
- The usual logical semantics is based on the  $\mathcal{D}$ -models of  $P$  and the models of  $P^*, T$ .
- The least  $\mathcal{D}$ -model of a formula  $Q$  is denoted by  $lm(Q, \mathcal{D})$ .
- A *solution* to a query  $G$  is a valuation  $v$  such that  $v(G) \subseteq lm(P, \mathcal{D})$ .

## Fixpoint Semantics

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- Based on one-step consequence operator  $T_P^{\mathcal{D}}$  (also called “immediate consequence operator”).
- Take as semantics  $lfp(T_P^{\mathcal{D}})$ , where:

$$T_P^{\mathcal{D}}(I) = \{p(\tilde{d}) \mid p(\tilde{x}) \leftarrow c, b_1, \dots, b_n \in P, a_i \in I, \\ \mathcal{D} \models v(c), v(\tilde{x}) = \tilde{d}, v(b_i) = a_i\}$$

- Theorems:
  1.  $T_P^{\mathcal{D}} \uparrow \omega = lfp(T_P^{\mathcal{D}})$
  2.  $lm(P, \mathcal{D}) = lfp(T_P^{\mathcal{D}})$

## Top–Down Operational Semantics (I)

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- General framework for operational semantics
- Formalized as a transition system on *states*
- State: a 3–tuple  $\langle A, C, S \rangle$ , or *fail*, where
  - ◇  $A$  is a multiset of atoms and constraints,
  - ◇  $C \cup S$  multiset of constraints,
  - ◇  $C$ , active constraints (awake)
  - ◇  $S$ , passive constraints (asleep)
- *Computation* and *Selection* rules depend on  $A$
- Transition system: parameterized by a predicate *consistent* and a function *infer*:
  - ◇ *consistent*( $C$ ) checks the consistency of a constraint store
  - ◇ Usually “*consistent*( $C$ ) iff  $\mathcal{D} \models \exists \tilde{c}$ ”, but sometimes “if  $\mathcal{D} \models \exists \tilde{c}$  then *consistent*( $C$ )”
  - ◇ *infer*( $C, S$ ) computes a new set of active and passive constraints

## Top-Down Operational Semantics (II)

- Transition  $r$ : computation step; rewriting using user predicates  
 $\langle A \cup a, C, S \rangle \rightarrow_r \langle A \cup B, C, S \cup (a = h) \rangle$   
 if  $h \leftarrow B \in P$ , and  $a$  and  $h$  have the same predicate symbol, or  
 $\langle A \cup a, C, S \rangle \rightarrow_r \text{fail}$   
 if there is no rule  $h \leftarrow B$  of  $P$  such that  $a$  and  $h$  have the same predicate symbol  
 ( $a = h$  is a set of argument-wise equations) if  $a$  is a predicate symbol selected by  
 the computation rule
- Transition  $c$ : selects constraints  
 $\langle A \cup c, C, S \rangle \rightarrow_c \langle A, C, S \cup c \rangle$   
 if  $c$  is a constraint selected by the computation rule
- Transition  $i$ : infers new constraints  
 $\langle A, C, S \rangle \rightarrow_i \langle A, C', S' \rangle$  if  $(C', S') = \text{infer}(C, S)$   
 ♦ In particular, may turn passive constraints into active ones
- Transition  $s$ : checks satisfiability  

$$\langle A, C, S \rangle \rightarrow_s \begin{cases} \langle A, C, S \rangle & \text{if } \text{consistent}(C) \\ \text{fail} & \text{if } \neg \text{consistent}(C) \end{cases}$$

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## Top-Down Operational Semantics (III)

- Initial state:  $\langle G, \emptyset, \emptyset \rangle$
- Derivation:  $\langle A_1, C_1, S_1 \rangle \rightarrow \dots \rightarrow \langle A_i, C_i, S_i \rangle \rightarrow \dots$
- Final state:  $E \rightarrow E$
- *Successful derivation*: final state  $\langle \emptyset, C, S \rangle$
- A derivation *flounders* if finite and the final state is  $\langle A, C, S \rangle$  with  $A \neq \emptyset$
- A derivation is *failed* if it is finite and the final state is fail
- Answer:  $\exists_{\tilde{x}} C \wedge S$ , where  $\tilde{x}$  are the variables in the initial goal
- A derivation is *fair* if it is failed or, for every  $i$  and every  $a \in A_i$ ,  $a$  is rewritten in a later transition
- A computation rule is fair if it gives rise only to fair derivations

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## Top-Down Operational Semantics (IV)

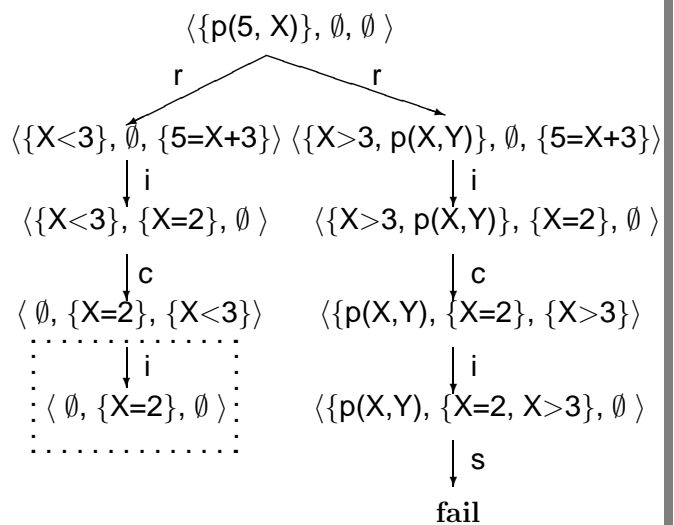
- *Computation tree* for goal  $G$  and program  $P$ :
  - ◇ Nodes labeled with states
  - ◇ Edges labeled with  $\rightarrow_r$ ,  $\rightarrow_c$ ,  $\rightarrow_i$  or  $\rightarrow_s$
  - ◇ Root labeled by  $\langle G, \emptyset, \emptyset \rangle$
  - ◇ All sons of a given node have the same label
  - ◇ Only one son with transitions  $\rightarrow_c$ ,  $\rightarrow_i$  or  $\rightarrow_s$
  - ◇ A son per program clause with transition  $\rightarrow_r$

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## Computation Tree: Example

- Consider the program
  - $p(X + 3, X) \leftarrow X < 3.$
  - $p(X + 3, X) \leftarrow X > 3, p(X, Y).$
 and the goal  $\leftarrow p(5, X)$

- A possible computation tree is:



- Dotted rectangle: previous state was final as well

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## Types of CLP( $\mathcal{X}$ ) Systems

- *Quick-checking* CLP( $\mathcal{X}$ ) system: its operational semantics can be described by  $\rightarrow_{ris} \equiv \rightarrow_r \rightarrow_i \rightarrow_s$  and  $\rightarrow_{cis} \equiv \rightarrow_c \rightarrow_i \rightarrow_s$
- I.e., always selects either an atom or a constraint, infers and checks consistency
- *Progressive* CLP system: for all  $\langle A, C, S \rangle$  with  $A \neq \emptyset$ , every derivation from that state either fails or contains a  $\rightarrow_r$  or  $\rightarrow_c$  transition
- *Ideal* CLP system:
  - ◊ Quick-checking
  - ◊ Progressive
  - ◊  $infer(C, S) = (C \cup S, \emptyset)$
  - ◊  $consistent(C)$  holds iff  $\mathcal{D} \models \exists c$

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## Soundness and Completeness Results

- **Success set:** the set of queries plus constraints which have a successful derivation in the program:
 
$$SS(P) = \{p(\tilde{x}) \leftarrow c \mid \langle p(\tilde{x}), \emptyset, \emptyset \rangle \rightarrow^* \langle \emptyset, c', c'' \rangle, \mathcal{D} \models c \leftrightarrow \exists_{-\tilde{x}} c' \wedge c''\}$$
- Consider a program  $P$  in the CLP language determined by a 4-tuple  $(\Sigma, \mathcal{D}, \mathcal{L}, \mathcal{T})$  and executing on an ideal CLP system. Then:
  1.  $[SS(P)]_{\mathcal{D}} = lm(P, \mathcal{D})$ , where
 
$$[SS(P)]_{\mathcal{D}} = \{v(a) \mid (a \leftarrow c) \in SS(P), \mathcal{D} \models v(c)\}$$
  2.  $SS(P) = lfp(S_P^{\mathcal{D}})$
  3. (Soundness) if the goal  $G$  has a successful derivation with answer constraint  $c$ , then  $P, \mathcal{T} \models c \rightarrow G$
  4. (Completeness) if  $P, \mathcal{T} \models c \rightarrow G$  then there are derivations for the goal  $G$  with answer constraints  $c_1, \dots, c_n$  such that  $\mathcal{T} \models c \rightarrow \bigvee_{i=1}^n c_i$
  5. Assume  $\mathcal{T}$  is satisfaction complete w.r.t.  $\mathcal{L}$ . Then the goal  $G$  is finitely failed for  $P$  iff  $P^*, \mathcal{T} \models \neg G$ .

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## Negation in CLP( $\mathcal{X}$ )

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- Most LP results can be lifted to CLP( $\mathcal{X}$ )
- In particular, negation as failure (à la SLDNF) is still valid using:
  - ◊ Satisfiability instead of unification
  - ◊ Variable elimination instead of groundness
- Added bonus: if the system is *solution compact*, then negated constraints can be expressed in terms of primitive constraints
- Less chances of a floundered / incorrect computation