Computational Logic

Fundamentals (of Definite Programs):

Syntax and *Semantics*

Towards Logic Programming

- Conclusion: resolution is a complete and effective deduction mechanism using:
  Horn clauses (related to "**D**efinite programs"),
  Linear, Input strategy
  Breadth-first exploration of the tree (or an equivalent approach)
  (possibly ordered clauses, but not required – see **S***election rule* later)
- Very close to what is generally referred to as SLD-resolution (see later)
- This allows to some extent realizing Greene's dream (within the theoretical limits of the formal method), and efficiently!

Towards Logic Programming (Contd.)

- Given these results, why not use logic as a general purpose *programming language*? [Kowalski 74]
- A “logic program” would have two interpretations:
  - *Declarative* (“LOGIC”): the logical reading (facts, statements, knowledge)
  - *Procedural* (“CONTROL”): what resolution does with the program
- ALGORITHM = LOGIC + CONTROL
- Specify these components separately
- Often, worrying about control is not needed at all (thanks to resolution)
- Control can be effectively provided through the ordering of the literals in the clauses

Towards Logic Programming: Another (more compact) Clausal Form

- All formulas are transformed into a set of *Clauses*.
  - A clause has the form: $\boxed{conc_1, ..., conc_m \leftarrow cond_1, ..., cond_n}$
    where $\boxed{\underbrace{conc_1, ..., conc_m}_{\text{“or”}}}$ $\boxed{\underbrace{cond_1, ..., cond_n}_{\text{“and”}}}$
    are literals, and are the *conclusions* and *conditions* of a rule:
    $\boxed{\underbrace{conc_1, ..., conc_m}_{\text{“conclusions”}} \leftarrow \underbrace{cond_1, ..., cond_n}_{\text{“conditions”}}}$
  - All variables are implicitly universally quantified: (if $X_1, ..., X_k$ are the variables)
    $\forall X_1, ..., X_k \quad conc_1 \vee ... \vee conc_m \leftarrow cond_1 \wedge ... \wedge cond_n$
- More compact than the traditional clausal form:
  - no connectives, just commas
  - no need to repeat negations: all negated atoms on one side, non-negated ones on the other
- A *Horn Clause* then has the form: $conc_1 \leftarrow cond_1, ..., cond_n$
  where n can be zero and possibly $conc_1$ empty.

Some Logic Programming Terminology – "Syntax" of Logic Programs

- *Definite Program:* a set of positive Horn clauses $head \leftarrow goal_1, ..., goal_n$
- The single *conclusion* is called the $\mathbf{head}$.
- The conditions are called "goals" or "procedure calls".
- $\mathbf{goal}_1\mathbf{,...,goal}_n$ ($n \geq 0$) is called the "body".
- if $n = 0$ the clause is called a "fact" (and the arrow is normally deleted)
- Otherwise it is called a "rule"
- *Query* (question): a negative Horn clause (a "headless" clause)
- A procedure is a set of rules and facts in which the heads have the same predicate symbol and arity.
- Terms in a goal are also called "arguments".

Some Logic Programming Terminology (Contd.)

- Examples:
  grandfather(X,Y) $\longleftarrow$ father (X,Z), mother(Z,Y).
  grandfather(X,Y) $\longleftarrow$.
  grandfather(X,Y).
  $\longleftarrow$ grandfather(X,Y).

LOGIC: Declarative "Reading" (Informal Semantics)

- A rule (has head and body)

$$head \leftarrow goal_1, ..., goal_n.$$

which contains variables $X_1, ..., X_k$ can be read as
for all $X_1, ..., X_k$:
"head" is true if "goal$_1$" and ... and "goal$_n$" are true

- A fact n=0 (has only head)

$$head.$$

for all $X_1, ..., X_k$: "head" is true (always)

- A query (the headless clause)

$$\leftarrow goal_1, ..., goal_n$$

can be read as:
for which $X_1, ..., X_k$ are "goal$_1$" and ... and "goal$_n$" true?

LOGIC: Declarative Semantics – Herbrand Base and Universe

- Given a first-order language L, with a non-empty set of variables, constants, function symbols, relation symbols, connectives, quantifiers, etc. and given a syntactic object A,

$$ground(A) = \{A\theta | \exists\theta \in Subst, var(A\theta) = \emptyset\}$$

 i.e. the set of all "ground instances" of A.

- Given L, U_L *(Herbrand universe)* is the set of all ground terms of L.
- B_L *(Herbrand Base)* is the set of all ground atoms of L.
- Similarly, for the language L_P associated with a given program P we define U_P, and B_P.
- Example:
 $P = \{ \quad p(f(X)) \leftarrow p(X). \quad p(a). \quad q(a). \quad q(b). \quad \}$
 $U_P = \{a, b, f(a), f(b), f(f(a)), f(f(b)), \ldots\}$
 $B_P = \{p(a), p(b), q(a), q(b), p(f(a)), p(f(b)), q(f(a)), \ldots\}$

Herbrand Interpretations and Models

- A *Herbrand Interpretation* is a subset of B_L, i.e. the set of all Herbrand interpretations $I_L = \wp(B_L)$.

  (Note that I_L forms a *complete lattice* under $\subseteq$ – important for fixpoint operations to be introduced later).

- Example: $P = \{ \quad p(f(X)) \leftarrow p(X). \quad p(a). \quad q(a). \quad q(b). \quad \}$
  $U_P = \{a, b, f(a), f(b), f(f(a)), f(f(b)), \ldots\}$
  $B_P = \{p(a), p(b), q(a), q(b), p(f(a)), p(f(b)), q(f(a)), \ldots\}$
  $I_P =$ *all subsets of* B_P

- A *Herbrand Model* is a Herbrand interpretation which contains all logical consequences of the program.

- The *Minimal Herbrand Model* H_P is the smallest Herbrand interpretation which contains all logical consequences of the program. (It is unique.)

- Example:
  $H_P = \{q(a), q(b), p(a), p(f(a)), p(f(f(a))), \ldots\}$

Declarative Semantics, Completeness, Correctness

- *Declarative semantics of a logic program* P:
  the set of ground facts which are logical consequences of the program (i.e., H_P).
  (Also called the “least model” semantics of P).
- *Intended meaning of a logic program* P:
  the set M of ground facts that the user expects to be logical consequences of the program.
- A logic program is *correct* if $H_P \subseteq M$.
- A logic program is *complete* if $M \subseteq H_P$.
- Example:

  father(john,peter).

  father(john,mary).

  mother(mary,mike).

  grandfather(X,Y) $\leftarrow$ father(X,Z), father(Z,Y).

  with the usual intended meaning is *correct* but *incomplete*.

CONTROL: Linear (Input) Resolution in this Clausal Form

We now turn to the *operational semantics* of logic programs,
given by a concrete operational procedure: *Linear (Input) Resolution*.

- Complementary literals:
  - in two different clauses
  - on different sides of $\leftarrow$
  - unifiable with unifier θ

$\mathbf{father(john,mary)}\leftarrow$
grandfather(X,Y) $\leftarrow$ $\mathbf{father(X,Z)}$, mother(Z,Y)

$\theta = \{X/john, Z/mary\}$

CONTROL: Linear (Input) Resolution in this Clausal Form (Contd.)

- Resolution step (linear, input, ...):
  - given a clause and a resolvent, we can build a new resolvent which follows from them by:
    - renaming apart the clause ("standardization apart" step)
    - putting *all* the conclusions to the left of the $\leftarrow$
    - putting *all* the conditions to the right of the $\leftarrow$
    - if there are complementary literals (unifying literals at different sides of the arrow in the two clauses), eliminating them and applying θ to the new resolvent
- LD-Resolution: linear (and input) resolution, applied to definite programs
  Note that then all resolvents are negative Horn clauses (like the query).

Example

- from

  $\text{father(john,peter)} \leftarrow$

  $\text{mother(mary,david)} \leftarrow$

  we can infer

  $\text{father(john,peter), mother(mary,david)} \leftarrow$

- from

  $\text{father(john,mary)} \leftarrow$

  $\text{grandfather(X,Y)} \leftarrow \text{father(X,Z), mother(Z,Y)}$

  we can infer

  $\text{grandfather(john,Y')} \leftarrow \text{mother(mary,Y')}$

CONTROL: A proof using LD-Resolution

- Prove "grandfather(john,david) $\leftarrow$" using the set of axioms:
  1. father(john,peter) $\leftarrow$
  2. father(john,mary) $\leftarrow$
  3. father(peter,mike) $\leftarrow$
  4. mother(mary,david) $\leftarrow$
  5. grandfather(L,M) $\leftarrow$ father (L,N), father(N,M)
  6. grandfather(X,Y) $\leftarrow$ father (X,Z), mother(Z,Y)
- We introduce the predicate to prove (negated!)

  7. $\leftarrow$ grandfather(john,david)
- We start resolution: e.g. 6 and 7

  8. $\leftarrow$ father(john,Z^1), mother(Z^1,david) X^1/john, Y^1/david
- using 2 and 8

  9. $\leftarrow$ mother(mary,david) Z^1/mary
- using 4 and 9

  $\leftarrow$

CONTROL: Rules and SLD-Resolution

- Two control-related issues are still left open in LD-resolution.
  Given a current resolvent R and a set of clauses K:
  - given a clause C in K, several of the literals in R may unify the non-negated a complementary literal in C
  - given a literal L in R, it may unify with complementary literals in several clauses in K
- A *Computation* (or *Selection* rule) is a function which, given a resolvent (and possibly the proof tree up to that point) returns (selects) a literal from it. This is the goal that will be used next in the resolution process.
- A *Search* rule is a function which, given a literal and a set of clauses (and possibly the proof tree up to that point), returns a clause from the set. This is the clause that will be used next in the resolution process.

CONTROL: Rules and SLD-Resolution (Contd.)

- SLD-resolution: Linear resolution for Definite programs with Selection rule.
- An SLD-resolution *method* is given by the combination of a *computation (or selection) rule* and a *search rule*.
- *Independence of the computation rule*: Completeness does not depend on the choice of the computation rule.
- Example: a "left-to-right" rule (as in ordered resolution) does not impair completeness – this coincides with the completeness result for ordered resolution.
- Fundamental result:
  "Declarative" semantics (H_P) $\equiv$ "operational" semantics (SLD-resolution)
  I.e., all the facts in H_P can be deduced using SLD-resolution.

CONTROL: Procedural reading of a logic program

- Given a rule

  $$head \leftarrow goal_1, ..., goal_n.$$

  it can be seen as a description of the goals the solver (resolution method) has to execute in order to solve "head"
- Possible, given computation and search rules.
- In general, "In order to solve 'head', solve 'goal$_1$' and ... and solve 'goal$_n$' "
- If ordered resolution is used (left-to-right computation rule), then read "In order to solve 'head', *first* solve 'goal$_1$' and *then* 'goal$_2$' and *then* ... and *finally* solve 'goal$_n$' "
- Thus the "control" part corresponding to the computation rule is often associated with the order of the goals in the body of a clause
- Another part (corresponding to the search rule) is often associated with the order of clauses

CONTROL: Procedural reading of a logic program (Contd.)

- Example – read "procedurally":

  father(john,peter).

  father(john,mary).

  father(peter,mike).

  father(X,Y) $\leftarrow$ mother(Z,Y), married(X,Z).

Towards a Fixpoint Semantics for LP – Fixpoint Basics

- A *fixpoint* for an operator $T : X \rightarrow X$ is an element of $x \in X$ such that $x = T(x)$.
- If X is a poset, T is monotonic if $\forall x, y \in X,\ x \leq y \Rightarrow T(x) \leq T(y)$
- If X is a complete lattice and T is monotonic the set of fixpoints of T is also a complete lattice [Tarski]
- The least element of the lattice is the *least fixpoint* of T, denoted $\mathit{lfp}\ (T)$
- Powers of a monotonic operator (successive applications):

  $T \uparrow 0(x) = x$
  $T \uparrow n(x) = T(T \uparrow (n-1)(x))$($n$ is a successor ordinal)
  $T \uparrow \omega(x) = \sqcup\{T \uparrow n(x) | n < \omega\}$

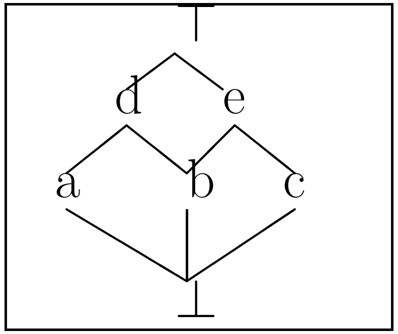
  We abbreviate $T \uparrow \alpha(\perp)$ as $T \uparrow \alpha$
- There is some ω such that $T \uparrow \omega = \mathit{lfp}\ T$. The sequence $T \uparrow 0, T \uparrow 1, ..., \mathit{lfp}\ T$ is the *Kleene sequence* for T
- In a finite lattice the Kleene sequence for a monotonic operator T is finite

Towards a Fixpoint Semantics for LP – Fixpoint Basics (Contd.)

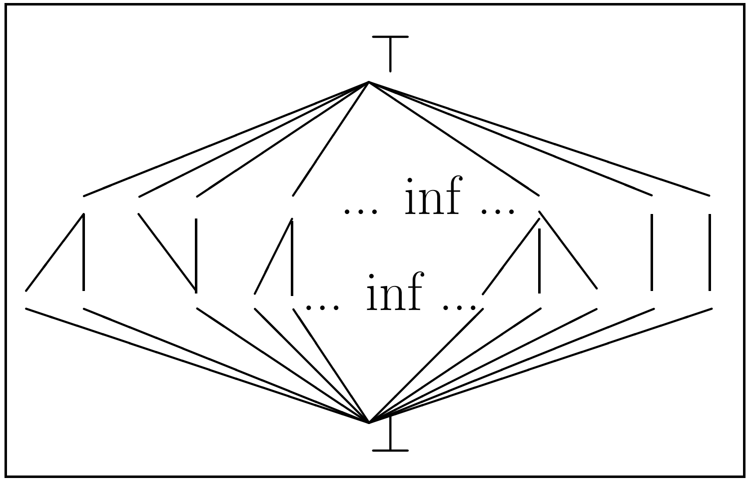
- A subset Y of a poset X is an (ascending) chain iff $\forall y, y' \in Y, y \leq y' \vee y' \leq y$
- A complete lattice X is *ascending chain finite* (or *Noetherian*) if all ascending chains are finite
- In an ascending chain finite lattice the Kleene sequence for a monotonic operator T is finite

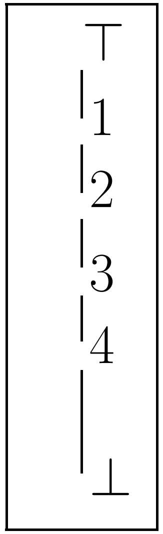
Lattice Structures

finite



finite_depth





ascending chain finite

A Fixpoint Semantics for Logic Programs, and Equivalences

- The *Immediate consequence operator* T_P is a mapping: $T_P : I_P \to I_P$ defined by:
  $T_P(I) = \{A \in B_P | \exists C \in ground(P), C = A \leftarrow L_1, ..., L_n$ and $L_1, \ldots L_n \in I\}$
  (in particular, if $(A \leftarrow) \in P$, then every element of $ground(A)$ is in $T_P(I)$, $\forall\ I$).
- T_P is monotonic, so it has a least fixpoint I^* so that $T_P(I^*) = I^*$, which can be obtained by applying T_P iteratively starting from the bottom element of the lattice (the empty interpretation)
- (Characterization Theorem) [Van Emden and Kowalski]
  A program P has a Herbrand model H_P such that :
  - ◇ H_P is the least Herbrand Model of P.
  - ◇ H_P is the least fixpoint of T_P (*lfp* T_P).
  - ◇ $H_P = T_P \uparrow \omega$.

  I.e., *least model semantics* (H_P) $\equiv$ *fixpoint semantics* (*lfp* T_P)
- Because it gives us some intuition on how to build H_P, the least fixpoint semantics can in some cases (e.g., finite models) also be an operational semantics (e.g., in *deductive databases*).

A Fixpoint Semantics for Logic Programs: Example

- Example:

$$
\begin{aligned}
P = \{\ & p(f(X)) \leftarrow p(X). \\
& p(a). \\
& q(a). \\
& q(b).\ \}
\end{aligned}
$$

$U_P = \{a, b, f(a), f(b), f(f(a)), f(f(b)), \ldots\}$
$B_P = \{p(a), p(b), q(a), q(b), p(f(a)), p(f(b)), q(f(a)), \ldots\}$
$I_P =$ *all subsets of* B
$H_P = \{q(a), q(b), p(a), p(f(a)), p(f(f(a))), \ldots\}$

$T_P \uparrow 0 = \{p(a), q(a), q(b)\}$
$T_P \uparrow 1 = \{p(a), q(a), q(b), p(f(a))\}$
$T_P \uparrow 2 = \{p(a), q(a), q(b), p(f(a)), p(f(f(a)))\}$
$\ldots$
$T_P \uparrow \omega = H_P$