Computational Logic
CLP Semantics and Fundamental Results
Constraint Domains

• Semantics parameterized by the constraint domain: CLP(\mathcal{X}), where \mathcal{X} ≡ (\Sigma, D, \mathcal{L}, \mathcal{T})

• Signature \Sigma: set of predicate and function symbols, together with their arity

• \mathcal{L} ⊆ \Sigma–formulae: constraints

• D is the set of actual elements in the domain

• \Sigma–structure D: gives the meaning of predicate and function symbols (and hence, constraints).

• \mathcal{T} a first–order theory (axiomatizes some properties of D)

• (D, \mathcal{L}) is a constraint domain

• Assumptions:
  ◦ \mathcal{L} built upon a first–order language
  ◦ = ∈ \Sigma is identity in D
  ◦ There are identically false and identically true constraints in \mathcal{L}
  ◦ \mathcal{L} is closed w.r.t. renaming, conjunction and existential quantification
Domains (I)

- $\Sigma = \{0, 1, +, *, =, <, \leq\}$, $D = \mathbb{R}$, $\mathcal{D}$ interprets $\Sigma$ as usual, $\mathbb{R} = (\mathcal{D}, \mathcal{L})$
  - Arithmetic over the reals
  - Eg.: $x^2 + 2xy < \frac{y}{x} \land x > 0$ ($\equiv xxx + xxy + xxy < y \land 0 < x$)
- Question: is $0$ needed? How can it be represented?

- Let us assume $\Sigma' = \{0, 1, +, =, <, \leq\}$, $\mathbb{R}_{Lin} = (\mathcal{D}', \mathcal{L}')$
  - Linear arithmetic
  - Eg.: $3x - y < 3$ ($\equiv x + x + x < 1 + 1 + 1 + y$)

- Let us assume $\Sigma'' = \{0, 1, +, =\}$, $\mathbb{R}_{LinEq} = (\mathcal{D}'', \mathcal{L}'')$
  - Linear equations
  - Eg.: $3x + y = 5 \land y = 2x$
Domains (II)

- $\Sigma = \{ <\text{constant and function symbols}>, = \}$
- $D = \{ \text{finite trees} \}$
- $D$ interprets $\Sigma$ as tree constructors
- Each $f \in \Sigma$ with arity $n$ maps $n$ trees to a tree with root labeled $f$ and whose subtrees are the arguments of the mapping
- Constraints: syntactic tree equality
- $\mathcal{FT} = (D, \mathcal{L})$
  - Constraints over the Herbrand domain
  - Eg.: $g(h(Z), Y) = g(Y, h(a))$
- $LP \equiv CLP(\mathcal{FT})$
Domains (III)

- $\Sigma = \{ <\text{constants}>, \lambda, \_, ::, =\}$
- $D = \{ \text{finite strings of constants} \}$
- $\mathcal{D}$ interprets $\cdot$ as string concatenation, $::$ as string length
  - Equations over strings of constants
  - Eg.: $X.A.X = X.A$

- $\Sigma = \{0, 1, \neg, \wedge, =\}$
- $D = \{\text{true, false}\}$
- $\mathcal{D}$ interprets symbols in $\Sigma$ as boolean functions
- $\text{BOOL} = (\mathcal{D}, \mathcal{L})$
  - Boolean constraints
  - Eg.: $\neg(x \wedge y) = 1$
CLP(\mathcal{X}) Programs

- Recall that:
  - $\Sigma$ is a set of predicate and function symbols
  - $\mathcal{L} \subseteq \Sigma$–formulae are the constraints
- $\Pi$: set of predicate symbols definable by a program
- Atom: $p(t_1, t_2, \ldots, t_n)$, where $t_1, t_2, \ldots, t_n$ are terms and $p \in \Pi$
- Primitive constraint: $p(t_1, t_2, \ldots, t_n)$, where $t_1, t_2, \ldots, t_n$ are terms and $p \in \Sigma$ is a predicate symbol
- Every constraint is a (first–order) formula built from primitive constraints
- The class of constraints will vary (generally only a subset of formulas are considered constraints)
- A CLP program is a collection of rules of the form $a \leftarrow b_1, \ldots, b_n$ where $a$ is an atom and the $b_i$’s are atoms or constraints
- A fact is a rule $a \leftarrow c$ where $c$ is a constraint
- A goal (or query) $G$ is a conjunction of constraints and atoms
Basic Operations on Constraints

- Constraint domains are expected to support some basic operations on constraints
  1. Consistency (or satisfiability) test: $\mathcal{D} \models \exists c$,  
  2. Implication or entailment: $\mathcal{D} \models c_0 \rightarrow c_1$,  
  3. Projection of a constraint $c_0$ onto variables $\tilde{x}$ to obtain a constraint $c_1$ such that $\mathcal{D} \models c_1 \iff \exists_{\tilde{x}} c_0$,  
  4. Detection of uniqueness of variable value: $\mathcal{D} \models c(x, \tilde{z}) \land c(y, \tilde{w}) \rightarrow x = y$

- Actually, only the first one is really required

- In actual implementations, some of these operations—in particular the test of consistency—may be incomplete

- Examples:
  - $x \times x < 0$ is inconsistent in $\mathbb{R}$ (because $\neg \exists x \in \mathbb{R} : x \times x < 0$)
  - $\mathcal{D} \models (x \land y = 1) \rightarrow (x \lor y = 1)$ in $\text{BOOL}$
  - In $\text{FT}$, the projection of $x = f(y) \land y = f(z)$ on $\{x, z\}$ is $x = f(f(z))$
  - In $\text{WE}$, $\mathcal{D} \models x.a \times x = x.a \land y.b \times y = y.b \rightarrow x = y$

- Prove the last assertion!
Properties of CLP Languages

- \( \mathcal{T} \) axiomatizes some of the properties of \( \mathcal{D} \)

- For a given \( \Sigma \), let \((\mathcal{D}, \mathcal{L})\) be a constraint domain with signature \( \Sigma \), and \( \mathcal{T} \) a \( \Sigma \)-theory.

- \( \mathcal{D} \) and \( \mathcal{T} \) correspond on \( \mathcal{L} \) if:
  - \( \mathcal{D} \) is a model of \( \mathcal{T} \), and
  - for every constraint \( c \in \mathcal{L} \), \( \mathcal{D} \models \exists c \iff \mathcal{T} \models \exists c \).

- \( \mathcal{T} \) is *satisfaction complete* with respect to \( \mathcal{L} \) if for every constraint \( c \in \mathcal{L} \), either \( \mathcal{T} \models \exists c \) or \( \mathcal{T} \models \lnot \exists c \).

- \((\mathcal{D}, \mathcal{L})\) is *solution compact* if
  \[
  \forall c \exists \{ c_i \}_{i \in I} : \mathcal{D} \models \forall \tilde{x} \lnot c(\tilde{x}) \iff \bigvee_{i \in I} c_i(\tilde{x})
  \]
  i.e., any negated constraint in \( \mathcal{L} \) can be expressed as a (in)finite disjunction of constraints
Solution Compactness

- Important to lift SLDNF results to CLP(\(\mathcal{A}\))
- We have to deal only with user predicates
- E.g.
  - \(x \not\geq y\) in CLP(\(\mathbb{R}\)) is \(x < y\)
  - \(x \neq y\) in CLP(\(\mathbb{R}\)) is \(x < y \lor y < x\)
  - \(\mathbb{R}_{Lin}\) with constraint \(x \neq \pi\) is not s.c.
- How can we express \(x \neq y\) in CLP(\(\mathcal{F}\mathcal{T}\))?
Two common logical semantics exist.

The first one interprets a rule

\[ p(\bar{x}) \leftarrow b_1, \ldots, b_n \]

as the logic formula

\[ \forall \bar{x}, \bar{y} \ p(\bar{x}) \lor \neg b_1 \lor \ldots \lor \neg b_n \]
Logical Semantics (II)

- The second one associates a logic formula to each predicate in \( \Pi \)
  - If the set of rules of \( P \) with \( p \) in the head is:
    
    \[
    \begin{align*}
    p(\tilde{x}) & \leftarrow B_1 \\
    p(\tilde{x}) & \leftarrow B_2 \\
    \vdots \\
    p(\tilde{x}) & \leftarrow B_n
    \end{align*}
    \]
    
    then the formula associated with \( p \) is:
    
    \[
    \forall \tilde{x} \ p(\tilde{x}) \iff \exists \tilde{y}_1 B_1 \lor \exists \tilde{y}_2 B_2 \lor \cdots \lor \exists \tilde{y}_n B_n
    \]

- If \( p \) does not occur in the head of a rule of \( P \), the formula is:
  
  \[ \forall \tilde{x} \neg p(\tilde{x}) \]

- The collection of all such formulas is the \textit{Clark completion} of \( P \) (denoted by \( P^* \))

- These two semantics differ on the treatment of the treatment of the negation
Logical Semantics (III)

- A *valuation* is a mapping from variables to $D$, and the natural extension which maps terms to $D$ and formulas to closed $L^*$–formulas.
- A $D$–interpretation of a formula is an interpretation of the formula with the same domain as $D$ and the same interpretation for the symbols in $\Sigma$ as $D$.
- It can be represented as a subset of $B_D$ where
  \[ B_D = \{ p(\tilde{d}) \mid p \in \Pi, \tilde{d} \in D^k \} \]
- A $D$–model of a closed formula is a $D$–interpretation which is a model of the formula.
- The usual logical semantics is based on the $D$–models of $P$ and the models of $P^*, \mathcal{T}$.
- The least $D$–model of a formula $Q$ is denoted by $lm(Q, D)$.
- A *solution* to a query $G$ is a valuation $v$ such that $v(G) \subseteq lm(P, D)$. 


Fixpoint Semantics

- Based on one-step consequence operator $T^D_P$ (also called “immediate consequence operator”).

- Take as semantics $\text{lfp}(T^D_P)$, where:

  $$T^D_P(I) = \{ p(\tilde{d}) \mid p(\tilde{x}) \leftarrow c, b_1, \ldots, b_n \in P, a_i \in I, \quad D \models v(c), v(\tilde{x}) = \tilde{d}, v(b_i) = a_i \}$$

- Theorems:
  1. $T^D_P \uparrow \omega = \text{lfp}(T^D_P)$
  2. $\text{lm}(P, D) = \text{lfp}(T^D_P)$
Top–Down Operational Semantics (I)

- General framework for operational semantics
- Formalized as a transition system on states
- State: a 3–tuple \( \langle A, C, S \rangle \), or \( \text{fail} \), where
  - \( A \) is a multiset of atoms and constraints,
  - \( C \cup S \) multiset of constraints,
  - \( C \), active constraints (awake)
  - \( S \), passive constraints (asleep)
- Computation and Selection rules depend on \( A \)
- Transition system: parameterized by a predicate \( \text{consistent} \) and a function \( \text{infer} \):
  - \( \text{consistent}(C) \) checks the consistency of a constraint store
  - Usually “\( \text{consistent}(C) \) iff \( D \models \exists c \)” but sometimes “if \( D \models \exists c \) then \( \text{consistent}(C) \)”
  - \( \text{infer}(C, S) \) computes a new set of active and passive constraints
Top–Down Operational Semantics (II)

- **Transition $r$:** computation step; rewriting using user predicates
  \[ \langle A \cup a, C, S \rangle \rightarrow_r \langle A \cup B, C, S \cup (a = h) \rangle \]
  if $h \leftarrow B \in P$, and $a$ and $h$ have the same predicate symbol, or
  \[ \langle A \cup a, C, S \rangle \rightarrow_r \text{fail} \]
  if there is no rule $h \leftarrow B$ of $P$ such that $a$ and $h$ have the same predicate symbol
  ($a = h$ is a set of argument–wise equations) if $a$ is a predicate symbol selected by the computation rule

- **Transition $c$:** selects constraints
  \[ \langle A \cup c, C, S \rangle \rightarrow_c \langle A, C, S \cup c \rangle \]
  if $c$ is a constraint selected by the computation rule

- **Transition $i$:** infers new constraints
  \[ \langle A, C, S \rangle \rightarrow_i \langle A, C', S' \rangle \text{ if } (C', S') = \text{infer}(C, S) \]
  ○ In particular, may turn passive constraints into active ones

- **Transition $s$:** checks satisfiability
  \[ \langle A, C, S \rangle \rightarrow_s \begin{cases} 
  \langle A, C', S' \rangle & \text{if consistent}(C) \\
  \text{fail} & \text{if } \neg \text{consistent}(C')
  \end{cases} \]
Top–Down Operational Semantics (III)

- Initial state: \( \langle G, \emptyset, \emptyset \rangle \)
- Derivation: \( \langle A_1, C_1, S_1 \rangle \rightarrow \ldots \rightarrow \langle A_i, C_i, S_i \rangle \rightarrow \ldots \)
- Final state: \( E \rightarrow E \)
- **Successful derivation**: final state \( \langle \emptyset, C, S \rangle \)
- A derivation **flounders** if finite and the final state is \( \langle A, C, S \rangle \) with \( A \neq \emptyset \)
- A derivation is **failed** if it is finite and the final state is fail
- Answer: \( \exists \tilde{x} C \land S \), where \( \tilde{x} \) are the variables in the initial goal
- A derivation is **fair** if it is failed or, for every \( i \) and every \( a \in A_i, a \) is rewritten in a later transition
- A computation rule is fair if it gives rise only to fair derivations
• *Computation tree* for goal $G$ and program $P$:

  ◆ Nodes labeled with states
  ◆ Edges labeled with $\rightarrow_r$, $\rightarrow_c$, $\rightarrow_i$ or $\rightarrow_s$
  ◆ Root labeled by $\langle G, \emptyset, \emptyset \rangle$
  ◆ All sons of a given node have the same label
  ◆ Only one son with transitions $\rightarrow_c$, $\rightarrow_i$ or $\rightarrow_s$
  ◆ A son per program clause with transition $\rightarrow_r$
Computation Tree: Example

- Consider the program
  \[ p(X + 3, X) \leftarrow X < 3. \]
  \[ p(X + 3, X) \leftarrow X > 3, p(X, Y). \]
  and the goal \( \leftarrow p(5, X) \)

- A possible computation tree is:

- Dotted rectangle: previous state was final as well
Types of CLP(\(\mathcal{X}\)) Systems

- **Quick-checking** CLP(\(\mathcal{X}\)) system: its operational semantics can be described by 
  \[ \rightarrow_{ris} \equiv \rightarrow_r i \rightarrow_s \text{ and } \rightarrow_{cis} \equiv \rightarrow_c i \rightarrow_s \]

- I.e., always selects either an atom or a constraint, infers and checks consistency

- **Progressive** CLP system: for all \(\langle A, C, S \rangle\) with \(A \neq \emptyset\), every derivation from that state either fails or contains a \(\rightarrow_r\) or \(\rightarrow_c\) transition

- **Ideal** CLP system:
  - Quick-checking
  - Progressive
  - \(\text{infer}(C, S) = (C \cup S, \emptyset)\)
  - \(\text{consistent}(C)\) holds iff \(\mathcal{D} \models \exists c\)
Soundness and Completeness Results

- **Success set**: the set of queries plus constraints which have a successful derivation in the program:
  \[ SS(P) = \{ p(\tilde{x}) \leftarrow c \mid \langle p(\tilde{x}), \emptyset, \emptyset \rangle \rightarrow^* \langle \emptyset, c', c'' \rangle, \mathcal{D} \models c \leftrightarrow \exists_{\tilde{x}} c' \land c'' \} \]

- Consider a program \( P \) in the CLP language determined by a 4–tuple \( (\Sigma, \mathcal{D}, \mathcal{L}, \mathcal{T}) \) and executing on an **ideal** CLP system. Then:
  1. \( [SS(P)]_\mathcal{D} = lm(P, \mathcal{D}) \), where
     \[ [SS(P)]_\mathcal{D} = \{ v(a) \mid (a \leftarrow c) \in SS(P), \mathcal{D} \models v(c) \} \]
  2. \( SS(P) = lfp(S^P_\mathcal{D}) \)
  3. (Soundness) if the goal \( G \) has a successful derivation with answer constraint \( c \), then \( P, \mathcal{T} \models c \rightarrow G \)
  4. (Completeness) if \( P, \mathcal{T} \models c \rightarrow G \) then there are derivations for the goal \( G \) with answer constraints \( c_1, \ldots, c_n \) such that \( \mathcal{T} \models c \rightarrow \bigvee_{i=1}^n c_i \)
  5. Assume \( \mathcal{T} \) is satisfaction complete w.r.t. \( \mathcal{L} \). Then the goal \( G \) is finitely failed for \( P \) iff \( P^*, \mathcal{T} \models \neg G \).
Negation in CLP(\(\mathcal{X}\))

- Most LP results can be lifted to CLP(\(\mathcal{X}\))
- In particular, negation as failure (à la SLDNF) is still valid using:
  - Satisfiability instead of unification
  - Variable elimination instead of groundness
- Added bonus: if the system is solution compact, then negated constraints can be expressed in terms of primitive constraints
- Less chances of a floundered / incorrect computation