Computational Logic

Logic Programming:

Model and Fixpoint Semantics
Towards the Model and Fixpoint Semantics

- We have seen previously the operational semantics (SLD-resolution).
- We now present the (declarative) **Model Semantics**:
  - We define our semantic *domain* (Herbrand interpretations).
  - We introduce the Minimal Herbrand Model.
- And the (also declarative) **Fixpoint Semantics**.
  - We recall some basic fixpoint theory.
  - Present the $T_P$ operator and the classic fixpoint semantics.
Declarative Semantics – Herbrand Base and Universe

- Given a first-order language $L$, with a non-empty set of variables, constants, function symbols, relation symbols, connectives, quantifiers, etc. and given a syntactic object $A$,

$$\text{ground}(A) = \{ A\theta | \exists \theta \in \text{Subst}, \text{var}(A\theta) = \emptyset \}$$

i.e. the set of all “ground instances” of $A$.

- Given $L$, $U_L$ (Herbrand universe) is the set of all ground terms of $L$.

- $B_L$ (Herbrand Base) is the set of all ground atoms of $L$.

- Similarly, for the language $L_P$ associated with a given program $P$ we define $U_P$, and $B_P$. 
Declarative Semantics – Herbrand Base and Universe (example)

- **Program:**
  
  \[
P = \{ p(X) \leftarrow q(X).
  
  p(a).
  
  p(b).
  
  q(c). \}
  
  \]

- **Herbrand universe:**
  
  \[U_P = \{a, b, c\}\]

- **Herbrand base:**
  
  \[B_P = \{p(a), p(b), p(c), q(a), q(b), q(c)\}\]
Declarative Semantics – Herbrand Base and Universe (example)

- **Program:**

  \[ P = \{ p(f(X)) \leftarrow p(X). \\
  \quad p(a). \\
  \quad q(a). \\
  \quad q(b). \} \]

- **Herbrand universe:**

  \[ U_P = \{ a, b, f(a), f(b), f(f(a)), f(f(b)), \ldots \} \]

- **Herbrand base:**

  \[ B_P = \{ p(a), p(b), q(a), q(b), p(f(a)), p(f(b)), q(f(a)), \ldots \} \]
Herbrand Interpretations and Models

- A *Herbrand Interpretation* is a subset of $B_L$, i.e. the set of all Herbrand interpretations $I_L = \wp(B_L)$.
  (Note that $I_L$ forms a *complete lattice* under $\subseteq$ – important for fixpoint operations to be introduced later).

- A *Herbrand Model* is a Herbrand interpretation which contains all logical consequences of the program.

- The *Minimal Herbrand Model* $H_P$ is the smallest Herbrand interpretation which contains all logical consequences of the program. (Theorem: it is unique.)
Declarative Semantics – Herbrand Model (example)

- **Program:**
  \[
P = \{ \ p(X) \leftarrow q(X). \\
p(a). \\
p(b). \\
q(c). \ \}
\]

- **Herbrand universe:**
  \[
  U_P = \{a, b, c\}
\]

- **Herbrand base:**
  \[
  B_P = \{p(a), p(b), p(c), q(a), q(b), q(c)\}
  \]

- **All possible interpretations:**
  \[
  I_P = \text{all subsets of } B_P
  \]

- **Herbrand model:**
  \[
  H_P = \{p(a), p(b), q(c), p(c)\}
  \]
Declarative Semantics – Herbrand Base and Universe (example)

• Program:

\[ P = \{ p(f(X)) \leftarrow p(X). \\
p(a). \\
q(a). \\
q(b). \} \]

• Herbrand universe:

\[ U_P = \{ a, b, f(a), f(b), f(f(a)), f(f(b)), \ldots \} \]

• Herbrand base:

\[ B_P = \{ p(a), p(b), q(a), q(b), p(f(a)), p(f(b)), q(f(a)), \ldots \} \]

• All possible interpretations:

\[ I_P = \text{all subsets of } B_P \]

• Herbrand model:

\[ H_P = \{ p(a), q(a), q(b), p(f(a)), p(f(f(a))), \ldots \} \]
Declarative Semantics, Completeness, Correctness

- **Declarative semantics of a logic program** $P$: the set of ground facts which are logical consequences of the program (i.e., $H_P$). (I.e., the *Minimal Herbrand* model (or “least model”) of $P$).

- **Intended meaning of a logic program** $P$: the set $I$ of ground facts that the user expects to be logical consequences of the program.

- A logic program is *correct* if $H_P \subseteq I$.

- A logic program is *complete* if $I \subseteq H_P$.

- Example:
  - father(john,peter).
  - father(john,mary).
  - mother(mary,mike).
  - grandfather(X,Y) ← father(X,Z), father(Z,Y).

  with the usual intended meaning is *correct* but *incomplete*. 
Towards a Fixpoint Semantics for LP – Fixpoint Basics

• A fixpoint for an operator $T : X \to X$ is an element of $x \in X$ such that $x = T(x)$.

• If $X$ is a poset, $T$ is monotonic if $\forall x, y \in X, x \leq y \Rightarrow T(x) \leq T(y)$

• If $X$ is a complete lattice and $T$ is monotonic the set of fixpoints of $T$ is also a complete lattice [Tarski]

• The least element of the lattice is the least fixpoint of $T$, denoted $\text{lfp } T$

• Powers of a monotonic operator (successive applications):

$$T \uparrow 0(x) = x$$
$$T \uparrow n(x) = T(T \uparrow (n - 1)(x)) (n \text{ is a successor ordinal})$$
$$T \uparrow \omega(x) = \bigsqcup \{ T \uparrow n(x) | n < \omega \}$$

We abbreviate $T \uparrow \alpha(\bot)$ as $T \uparrow \alpha$

• There is some $\omega$ such that $T \uparrow \omega = \text{lfp } T$. The sequence $T \uparrow 0, T \uparrow 1, ..., \text{lfp } T$ is the Kleene sequence for $T$

• In a finite lattice the Kleene sequence for a monotonic operator $T$ is finite
Towards a Fixpoint Semantics for LP – Fixpoint Basics (Contd.)

• A subset $Y$ of a poset $X$ is an (ascending) chain iff $\forall y, y' \in Y, y \leq y' \lor y' \leq y$

• A complete lattice $X$ is ascending chain finite (or Noetherian) if all ascending chains are finite

• In an ascending chain finite lattice the Kleene sequence for a monotonic operator $T$ is finite
Lattice Structures

finite

finite_depth

ascending chain finite
A Fixpoint Semantics for Logic Programs

- **Semantic domain**: $I_L = \wp(B_L)$.

- I.e., the elements of the semantic domain and interpretations (subsets of the Herbrand base).

- **Semantic operator** (defined on programs):
  the *immediate consequences operator*, $T_P$:
  
  - $T_P$ is a mapping: $T_P : I_P \rightarrow I_P$ defined by:
    
    $$T_P(I) = \{ A \in B_P \mid \exists C \in ground(P), C = A \leftarrow L_1, \ldots, L_n \text{ and } L_1, \ldots L_n \in I \}$$
    
    (in particular, if $(A \leftarrow) \in P$, then every element of $ground(A)$ is in $T_P(I)$, $\forall I$).

- $T_P$ is monotonic, so:
  
  - it has a least fixpoint $I^*$ so that $T_P(I^*) = I^*$,
  
  - this fixpoint can be obtained by applying $T_P$ iteratively starting from the bottom element of the lattice (the empty interpretation).
A Fixpoint Semantics for Logic Programs: Example 1 (finite)

\[ P = \{ p(X, a) \leftarrow q(X). \]
\[ p(X, Y) \leftarrow q(X), r(Y). \]
\[ q(a). \quad r(b). \]
\[ q(b). \quad r(c). \} \]

\[ U_P = \{ a, b, c \} \]

\[ B_P = \{ p(a, a), p(a, b), p(a, c), p(b, a), p(b, b), p(b, c), p(c, a), p(c, b), p(c, c), \]
\[ q(a), q(b), q(c), \]
\[ r(a), r(b), r(c) \} \]

\[ I_P = \text{all subsets of } B_P \]

\[ H_P = \{ q(a), q(b), r(b), r(c), p(a, a), p(b, a), p(a, b), p(b, b), p(a, c), p(b, c) \} \]

\[ T_P \uparrow 0 = \{ q(a), q(b), r(b), r(c) \} \]
\[ T_P \uparrow 1 = \{ q(a), q(b), r(b), r(c) \} \cup \{ p(a, a), p(b, a), p(a, b), p(b, b), p(a, c), p(b, c) \} \]
\[ T_P \uparrow 2 = T_P \uparrow 1 = \text{lfp}(T_P) = H_P \]
A Fixpoint Semantics for Logic Programs: Example 2 (infinite)

\[ P = \{ p(f(X)) \leftarrow p(X). \]
\[ \quad p(a). \]
\[ \quad q(a). \]
\[ \quad q(b). \} \]

\[ U_P = \{ a, b, f(a), f(b), f(f(a)), f(f(b)), \ldots \} \]
\[ B_P = \{ p(a), p(b), q(a), q(b), p(f(a)), p(f(b)), q(f(a)), \ldots \} \]
\[ I_P = \text{all subsets of } B_P \]
\[ H_P = \{ q(a), q(b), p(a) \} \cup \{ p(f^n(a)) \mid n \in \mathbb{N} \} \]

where we define \( f^n(a) \) to be \( f \) nested \( n \) times and then applied to \( a. \)

(i.e., \( q(a), q(b), p(a), p(f(a)), p(f(f(a))), p(f(f(f(a)))), \ldots \))

\[ T_P \uparrow 0 = \{ p(a), q(a), q(b) \} \]
\[ T_P \uparrow 1 = \{ p(a), q(a), q(b), p(f(a)) \} \]
\[ T_P \uparrow 2 = \{ p(a), q(a), q(b), p(f(a)), p(f(f(a))) \} \]
\[ \ldots \]
\[ T_P \uparrow \omega = H_P \]
A Fixpoint Semantics for Logic Programs: Example 3 (infinite)

- Example:

\[ P = \{ \text{nat}(0). \text{nat}(s(X)) ← \text{nat}(X). \text{sum}(0, X, X). \text{sum}(s(X), Y, s(Z)) ← \text{sum}(X, Y, Z). \} \]

\[ U_P = \{0\} \cup \{s(x) \mid x \in U_P\} \]

(i.e., \{0, s(0), s(s(0)), s(s(s(0))), ...\}).

\[ B_P = \{\text{nat}(x) \mid x \in U_P\} \cup \{\text{sum}(x, y, z) \mid x, y, z \in U_P\} \]

(i.e., \{\text{nat}(0), \text{nat}(s(0)), \text{nat}(s(s(0))), ...\} \cup \{\text{sum}(0, 0, 0), \text{sum}(s(0), 0, 0), \text{sum}(0, s(0), 0), \text{sum}(0, 0, s(0)), ...\}).
Constructing the least fixpoint of the $T_P$ operator:

\[
\begin{align*}
T_P \uparrow 0 &= \{\text{nat}(0)\} \cup \{\text{sum}(0, x, x) \mid x \in U_P\} \\
T_P \uparrow 1 &= T_P \uparrow 0 \cup \{\text{nat}(s(0))\} \\
&\quad \cup \{\text{sum}(s(0), y, s(y)) \mid y \in U_P\} \\
T_P \uparrow 2 &= T_P \uparrow 1 \cup \{\text{nat}(s(s(0)))\} \\
&\quad \cup \{\text{sum}(s(s(0)), y, s(s(y))) \mid y \in U_P\} \\
T_P \uparrow 3 &= T_P \uparrow 2 \cup \{\text{nat}(s(s(s(0))))\} \\
&\quad \cup \{\text{sum}(s(s(s(0))), y, s(s(s(y)))) \mid y \in U_P\} \\
&\quad \ldots \\
T_P \uparrow \omega &= \{\text{nat}(x) \mid x \in U_P\} \cup \\
&\quad \{\text{sum}(s^n(0), y, s^n(y)) \mid y \in U_P \land n \in \mathbb{N}\}
\end{align*}
\]

where we define $s^x(y)$ to be $s$ nested $x$ times and then applied to $y$. 
Semantics – Equivalences

• (Characterization Theorem) [Van Emden and Kowalski]
  A program $P$ has a Herbrand model $H_P$ such that :
  
  ◦ $H_P$ is the least Herbrand Model of $P$.
  ◦ $H_P$ is the least fixpoint of $T_P$ ($lfp T_P$).
  ◦ $H_P = T_P \uparrow \omega$.
  
  I.e., least model semantics ($H_P$) $\equiv$ fixpoint semantics ($lfp T_P$)

• In addition, there is also an equivalence with the operational semantics (SLD-resolution):
  
  ◦ SLD-resolution answers “yes” to $a \in B_P \iff a \in H_P$.

• Because it gives us a way to directly build $H_P$ (for finite models), the least fixpoint semantics can in some cases also be an operational semantics (e.g., for datalog in deductive databases).