Computational Logic

Logic Programming:

*Model and Fixpoint Semantics*
Towards the Model and Fixpoint Semantics

- We have seen previously the operational semantics (SLD-resolution).
- We now present the (declarative) Model Semantics:
  - We define our semantic domain (Herbrand interpretations).
  - We introduce the Minimal Herbrand Model.
- And the (also declarative) Fixpoint Semantics.
  - We recall some basic fixpoint theory.
  - Present the $T_P$ operator and the classic fixpoint semantics.
Declarative Semantics – Herbrand Base and Universe

- Given a first-order language \( L \), with a non-empty set of variables, constants, function symbols, relation symbols, connectives, quantifiers, etc. and given a syntactic object \( A \),

\[
\text{ground}(A) = \{ A\theta | \exists \theta \in \text{Subst}, \text{var}(A\theta) = \emptyset \}
\]

i.e. the set of all “ground instances” of \( A \).

- Given \( L \), \( U_L \) (*Herbrand universe*) is the set of all ground terms of \( L \).

- \( B_L \) (*Herbrand Base*) is the set of all ground atoms of \( L \).

- Similarly, for the language \( L_P \) associated with a given program \( P \) we define \( U_P \), and \( B_P \).
Program:

\[ P = \{ \ p(f(X)) \leftarrow p(X) . \ 
    p(a) . \ 
    q(a) . \ 
    q(b) . \ \} \]

Herbrand universe:

\[ U_P = \{ a, b, f(a), f(b), f(f(a)), f(f(b)), \ldots \} \]

Herbrand base:

\[ B_P = \{ p(a), p(b), q(a), q(b), p(f(a)), p(f(b)), q(f(a)), \ldots \} \]
Herbrand Interpretations and Models

• A Herbrand Interpretation is a subset of $B_L$, i.e. the set of all Herbrand interpretations $I_L = \mathcal{P}(B_L)$.

(Note that $I_L$ forms a complete lattice under $\subseteq$ – important for fixpoint operations to be introduced later).

• In previous example: $P = \{ p(f(X)) \leftarrow p(X). \ p(a). \ q(a). \ q(b). \}$
$U_P = \{ a, b, f(a), f(b), f(f(a)), f(f(b)), \ldots \}$
$B_P = \{ p(a), p(b), q(a), q(b), p(f(a)), p(f(b)), q(f(a)), \ldots \}$
$I_P = all \ subsets \ of \ B_P$

• A Herbrand Model is a Herbrand interpretation which contains all logical consequences of the program.

• The Minimal Herbrand Model $H_P$ is the smallest Herbrand interpretation which contains all logical consequences of the program. (Theorem: it is unique.)

• Example:
$H_P = \{ q(a), q(b), p(a), p(f(a)), p(f(f(a))), \ldots \}$
Declarative Semantics, Completeness, Correctness

- **Declarative semantics of a logic program** $P$:
  the set of ground facts which are logical consequences of the program (i.e., $H_P$).
  (I.e., the *Minimal Herbrand* model (or “least model”) of $P$).

- **Intended meaning of a logic program** $P$:
  the set $I$ of ground facts that the user expects to be logical consequences of the program.

- A logic program is **correct** if $H_P \subseteq I$.

- A logic program is **complete** if $I \subseteq H_P$.

- Example:
  
  father(john,peter).
  father(john,mary).
  mother(mary,mike).
  grandfather(X,Y) ← father(X,Z), father(Z,Y).

  with the usual intended meaning is **correct** but **incomplete**.
Towards a Fixpoint Semantics for LP – Fixpoint Basics

- A *fixpoint* for an operator $T : X \rightarrow X$ is an element of $x \in X$ such that $x = T(x)$.

- If $X$ is a poset, $T$ is monotonic if $\forall x, y \in X, x \leq y \Rightarrow T(x) \leq T(y)$

- If $X$ is a complete lattice and $T$ is monotonic the set of fixpoints of $T$ is also a complete lattice [Tarski]

- The least element of the lattice is the *least fixpoint* of $T$, denoted $lfp(T)$

- Powers of a monotonic operator (successive applications):
  
  $$T \uparrow 0(x) = x$$
  $$T \uparrow n(x) = T(T \uparrow (n - 1)(x)) \text{ (} n \text{ is a successor ordinal)}$$
  $$T \uparrow \omega(x) = \bigsqcup \{T \uparrow n(x) \mid n < \omega\}$$

  We abbreviate $T \uparrow \alpha(\bot)$ as $T \uparrow \alpha$

- There is some $\omega$ such that $T \uparrow \omega = lfp(T)$. The sequence $T \uparrow 0, T \uparrow 1, ..., lfp(T)$ is the *Kleene sequence* for $T$

- In a finite lattice the Kleene sequence for a monotonic operator $T$ is finite
Towards a Fixpoint Semantics for LP – Fixpoint Basics (Contd.)

- A subset $Y$ of a poset $X$ is an (ascending) chain iff $\forall y, y' \in Y, y \leq y' \lor y' \leq y$
- A complete lattice $X$ is *ascending chain finite* (or *Noetherian*) if all ascending chains are finite
- In an ascending chain finite lattice the Kleene sequence for a monotonic operator $T$ is finite
Lattice Structures

finite

finite_depth

ascending chain finite
A Fixpoint Semantics for Logic Programs

- **Semantic domain:** \( I_L = \wp(B_L) \).
- I.e., the elements of the semantic domain and *interpretations* (subsets of the Herbrand base).
- **Semantic operator** (defined on programs): the *immediate consequences operator*, \( T_P \):
  - \( T_P \) is a mapping: \( T_P : I_P \rightarrow I_P \) defined by:
    \[
    T_P(I) = \{ A \in B_P \mid \exists C \in \text{ground}(P), C = A \leftarrow L_1, \ldots, L_n \text{ and } L_1, \ldots, L_n \in I \}
    \]
    (in particular, if \((A \leftarrow) \in P\), then every element of \(\text{ground}(A)\) is in \(T_P(I), \forall I\)).
- \( T_P \) is monotonic, so:
  - it has a least fixpoint \( I^* \) so that \( T_P(I^*) = I^* \),
  - this fixpoint can be obtained by applying \( T_P \) iteratively starting from the bottom element of the lattice (the empty interpretation).
A Fixpoint Semantics for Logic Programs: Example 1 (finite)

\[ P = \{ p(X, a) \leftarrow q(X). \]
\[ p(X, Y) \leftarrow q(X), r(Y). \]
\[ q(a). \quad r(b). \]
\[ q(b). \quad r(c). \} \]

\[ U_P = \{a, b, c\} \]

\[ B_P = \{ p(a, a), p(a, b), p(a, c), p(b, a), p(b, b), p(b, c), p(c, a), p(c, b), p(c, c), \]
\[ q(a), q(b), q(c), \]
\[ r(a), r(b), r(c)\} \]

\[ I_P = \text{all subsets of } B_P \]

\[ H_P = \{q(a), q(b), r(b), r(c), p(a, a), p(b, b), p(a, b), p(b, a), p(a, c), p(b, c)\} \]

\[ T_P \uparrow 0 = \{q(a), q(b), r(b), r(c)\} \]
\[ T_P \uparrow 1 = \{q(a), q(b), r(b), r(c)\} \cup \{p(a, a), p(b, a), p(a, b), p(b, b), p(a, c), p(b, c)\} \]
\[ T_P \uparrow 2 = T_P \uparrow 1 = \text{lfp}(T_P) = H_P \]
A Fixpoint Semantics for Logic Programs: Example 2 (infinite)

\[
P = \{ \ p(f(X)) \leftarrow p(X). \ 
\quad p(a). 
\quad q(a). 
\quad q(b). \ \} 
\]

\[
U_P = \{ a, b, f(a), f(b), f(f(a)), f(f(b)), \ldots \} 
\]

\[
B_P = \{ p(a), p(b), q(a), q(b), p(f(a)), p(f(b)), q(f(a)), \ldots \} 
\]

\[
I_P = \text{all subsets of } B_P 
\]

\[
H_P = \{ q(a), q(b), p(a) \} \cup \{ p(f^n(a)) \mid n \in \mathbb{N} \} 
\]

where we define \( f^n(a) \) to be \( f \) nested \( n \) times and then applied to \( a \).
(i.e., \( q(a), q(b), p(a), p(f(a)), p(f(f(a))), p(f(f(f(a)))), \ldots \))

\[
T_P \uparrow 0 = \{ p(a), q(a), q(b) \} 
\]

\[
T_P \uparrow 1 = \{ p(a), q(a), q(b), p(f(a)) \} 
\]

\[
T_P \uparrow 2 = \{ p(a), q(a), q(b), p(f(a)), p(f(f(a))) \} 
\]

\[
\ldots 
\]

\[
T_P \uparrow \omega = H_P 
\]
A Fixpoint Semantics for Logic Programs: Example 3 (infinite)

- Example:

\[ P = \{ \text{nat}(0). \]
\[ \text{nat}(\text{s}(X)) \leftarrow \text{nat}(X). \]
\[ \text{sum}(0, X, X). \]
\[ \text{sum}(\text{s}(X), Y, \text{s}(Z)) \leftarrow \text{sum}(X, Y, Z). \} \]

\[ U_P = \{0\} \cup \{s(x) \mid x \in U_P\} \]

(i.e., \{0, s(0), s(s(0)), s(s(s(0))), ...\}).

\[ B_P = \{\text{nat}(x) \mid x \in U_P\} \cup \{\text{sum}(x, y, z) \mid x, y, z \in U_P\} \]

(i.e., \{\text{nat}(0), \text{nat}(s(0)), \text{nat}(s(s(0))), ...\} \cup
\{\text{sum}(0, 0, 0), \text{sum}(s(0), 0, 0), \text{sum}(0, s(0), 0), \text{sum}(0, 0, s(0)), ...\}).
Constructing the least fixpoint of the $T_P$ operator:

$$T_P \uparrow 0 = \{\text{nat}(0)\} \cup \{\text{sum}(0, x, x) \mid x \in U_P\}$$

$$T_P \uparrow 1 = T_P \uparrow 0 \cup \{\text{nat}(s(0))\}$$

$$\quad \cup \{\text{sum}(s(0), y, s(y)) \mid y \in U_P\}$$

$$T_P \uparrow 2 = T_P \uparrow 1 \cup \{\text{nat}(s(s(0)))\}$$

$$\quad \cup \{\text{sum}(s(s(0)), y, s(s(y))) \mid y \in U_P\}$$

$$T_P \uparrow 3 = T_P \uparrow 2 \cup \{\text{nat}(s(s(s(0))))\}$$

$$\quad \cup \{\text{sum}(s(s(s(0))), y, s(s(y))) \mid y \in U_P\}$$

... 

$$T_P \uparrow \omega = \{\text{nat}(x) \mid x \in U_P\} \cup$$

$$\quad \{\text{sum}(s^n(0), y, s^n(y)) \mid y \in U_P \land n \in \mathbb{N}\}$$

where we define $s^x(y)$ to be $s$ nested $x$ times and then applied to $y$. 
Semantics – Equivalences

• (Characterization Theorem) [Van Emden and Kowalski]
  A program $P$ has a Herbrand model $H_P$ such that:
  ◇ $H_P$ is the least Herbrand Model of $P$.
  ◇ $H_P$ is the least fixpoint of $T_P$ ($lfp \ T_P$).
  ◇ $H_P = T_P \uparrow \omega$.

I.e., least model semantics ($H_P$) $\equiv$ fixpoint semantics ($lfp \ T_P$)

• In addition, there is also an equivalence with the operational semantics
  (SLD-resolution):
  ◇ SLD-resolution answers “yes” to $a \in B_P$ $\iff$ $a \in H_P$.

• Because it gives us a way to directly build $H_P$ (for finite models), the least fixpoint
  semantics can in some cases also be an operational semantics (e.g., for $datalog$
  in deductive databases).