Computational Logic
Automated Deduction Fundamentals
Elements of First-Order Predicate Logic

First Order Language:

- An *alphabet* consists of the following classes of symbols:
  1. *variables* denoted by $X, Y, Z, Boo, \ldots$, (infinite)
  2. *constants* denoted by $1, a, boo, john, \ldots$
  3. *functors* denoted by $f, g, +, -, \ldots$
  4. *predicate symbols* denoted by $p, q, dog, \ldots$
  5. *connectives*, which are: $\neg$ (negation), $\lor$ (disjunction), $\land$ (conjunction), $\rightarrow$ (implication) and $\leftrightarrow$ (equivalence),
  6. *quantifiers*, which are: $\exists$ (there exists) and $\forall$ (for all),
  7. *parentheses*, which are: ( and ) and the *comma*, that is: “,”.

- Each functor and predicate symbol has a fixed *arity*, they are often represented in *Functor/Arity* form, e.g. f/3.
- A constant can be seen as a functor of arity 0.
- Propositions are represented by a predicate symbol of arity 0.
Important: Notation Convention Used

(A bit different from standard notational conventions in logic, but good for compatibility with LP systems)

- Variables: start with a capital letter or a “_” (X, Y, _a, _1)
- Atoms, functors, predicate symbols: start with a lower case letter or are enclosed in ’ ’ (f, g, a, 1, x, y, z, ‘X’, ’1’)
Terms and Atoms

We define by induction two classes of strings of symbols over a given alphabet.

- The class of *terms*:
  - a variable is a term,
  - a constant is a term,
  - if $f$ is an $n$-ary functor and $t_1, \ldots, t_n$ are terms then $f(t_1, \ldots, t_n)$ is a term.

- The class of *atoms* (different from LP!):
  - a proposition is an atom,
  - if $p$ is an $n$-ary pred. symbol and $t_1, \ldots, t_n$ are terms then $p(t_1, \ldots, t_n)$ is an atom,
  - true and false are atoms.

- The class of Well Formed Formulas (WFFs):
  - an atom is a WFF,
  - if $F$ and $G$ are WFFs then so are $\neg F, (F \lor G), (F \land G), (F \rightarrow G)$ and $(F \leftrightarrow G),$
  - if $F$ is a WFF and $X$ is a variable then $\exists X F$ and $\forall X F$ are WFF.

- Literal: positive or negative (non-negated or negated) atom.
Examples of Terms

- Given:
  - constants: $a, b, c, 1, \text{spot, john}$...
  - functors: $f/1, g/3, h/2, +/3$...
  - variables: $X, L, Y$...
- Correct: $\text{spot, } f(\text{john}), f(X), +(1,2,3), +(X,Y,L), f(f(\text{spot})), h(h(1,2),L)$
- Incorrect: $\text{spot}(X), +(1,2), g, f(f(h))$

Examples of Literals

- Given the elements above and:
  - predicate symbols: $\text{dog}/1, p/2, q/0, r/0, \text{barks}/1$...
- Correct: $q, r, \text{dog(\text{spot})}, p(X,f(\text{john}))$...
- Incorrect: $q(X), \text{barks}(f), \text{dog(\text{barks}(X))}$
Examples of WFFs

- Given the elements above

- Correct: \( q, q \rightarrow r, r \leftarrow q, \text{dog}(X) \leftarrow \text{barks}(X), \text{dog}(X), p(X,Y), \exists X (\text{dog}(X) \land \text{barks}(X) \land \neg q), \exists Y (\text{dog}(Y) \rightarrow \text{bark}(Y)) \)

- Incorrect: \( q \lor, \exists p \)
More about WFFs

- Allow us to represent knowledge and reason about it
  - Marcus was a man \( \text{man(marcus)} \)
  - Marcus was a pompeian \( \text{pompeian(marcus)} \)
  - All pompeians were romans \( \forall X \text{ pompeian}(X) \rightarrow \text{roman}(X) \)
  - Caesar was a ruler \( \text{ruler(caesar)} \)
  - All romans were loyal to Caesar or they hated him \( \forall X \text{ roman}(X) \rightarrow \text{loyalto}(X,\text{caesar}) \lor \text{hate}(X,\text{caesar}) \)
  - Everyone is loyal to someone \( \forall X \exists Y \text{ loyalto}(X,Y) \)

- We can now reason about this knowledge using standard deductive mechanisms.
- But there is in principle no guarantee that we will prove a given theorem.
Towards Efficient Automated Deduction

• *Automated deduction is search.*

• Complexity of search: directly dependent on branching factor at nodes (exponentially!).

• It is vital to cut down the branching factor:
  ◦ Canonical representation of nodes (allows identifying identical nodes).
  ◦ As few inference rules as possible.
Towards Efficient Automated Deduction (Contd.)

Clausal Form

- The complete set of logical operators ($\leftarrow$, $\land$, $\lor$, $\neg$, ...) is redundant.
- A minimal (canonical) form would be interesting.
- It would be interesting to separate the quantifiers from the rest of the formula so that they did not need to be considered.
- It would also be nice if the formula were flat (i.e. no parenthesis).
- Conjunctive normal form has these properties [Davis 1960].

Deduction Mechanism

- A good example:
  Resolution – only two inference rules (Resolution rule and Replacement rule).
Classical Clausal Form: Conjunctive Normal Form

- General formulas are converted to:
  - Set of *Clauses*.
  - Clauses are in a logical conjunction.
  - A clause is a disjunction of the form: \( \text{literal}_1 \lor \text{literal}_2 \lor \ldots \lor \text{literal}_n \)
  - The \( \text{literal}_i \) are negated or non-negated atoms.
  - All variables are implicitly universally quantified: i.e. if \( X_1, \ldots, X_k \) are the variables that appear in a clause it represents the formula:
    \[ \forall X_1, \ldots, X_k \ \text{literal}_1 \lor \text{literal}_2 \lor \ldots \lor \text{literal}_n \]

- Any formula can be converted to clausal form automatically by:
  1. Converting to Prenex form.
  2. Converting to conjunctive normal form (conjunction of disjunctions).
  3. Converting to Skolem form (eliminating existential quantifiers).
  4. Eliminating universal quantifiers.
  5. Separating conjunctions into clauses.

- The *unsatisfiability* of a system is preserved.
Substitutions

• A substitution is a finite mapping from variables to terms, written as 
  \( \theta = \{X_1/t_1, \ldots, X_n/t_n\} \) where
  ◦ the variables \( X_1, \ldots, X_n \) are different,
  ◦ for \( i = 1, \ldots, n \), \( X_i \not\equiv t_i \).

• A pair \( X_i/t_i \) is called a binding.

• \( \text{domain}(\theta) = \{X_1, \ldots, X_n\} \) and \( \text{range}(\theta) = \text{vars}(\{t_1, \ldots, t_n\}) \).

• If \( \text{range}(\theta) = \emptyset \) then \( \theta \) is called ground.

• If \( \theta \) is a bijective mapping from variables to variables then \( \theta \) is called a renaming.

• Examples:
  ◦ \( \theta_1 = \{X/f(A), Y/X, Z/h(b, Y), W/a\} \)
  ◦ \( \theta_2 = \{X/a, Y/a, Z/h(b, c), W/f(d)\} \) (ground)
  ◦ \( \theta_3 = \{X/A, Y/B, Z/C, W/D\} \) (renaming)
Substitutions (Contd.)

- Substitutions operate on expressions, i.e. a term, a sequence of literals or a clause, denoted by $E$.

- The application of $\theta$ to $E$ (denoted $E\theta$) is obtained by simultaneously replacing each occurrence in $E$ of $X_i$ by $t_i$, $X_i/t_i \in \theta$.

- The resulting expression $E\theta$ is called an instance of $E$.

- If $\theta$ is a renaming then $E\theta$ is called a variant of $E$.

- Example:

  \[
  \theta_1 = \{X/f(A), Y/X, Z/h(b,Y), W/a\}
  \]

  \[
  p(X,Y,X) \theta_1 = p(f(A), X, f(A))
  \]
Composition of Substitutions

- Given $\theta = \{X_1/t_1, ..., X_n/t_n\}$ and $\eta = \{Y_1/s_1, ..., Y_m/s_m\}$ their composition $\theta \eta$ is defined by removing from the set
  \[
  \{X_1/t_1 \eta, ..., X_n/t_n \eta, Y_1/s_1, ..., Y_m/s_m\}
  \]
  those pairs $X_i/t_i \eta$ for which $X_i \equiv t_i \eta$, as well as those pairs $Y_i/s_i$ for which $Y_i \in \{X_1, ..., X_n\}$.

- Example: if $\theta = \{X/3, Y/f(X, 1)\}$ and $\eta = \{X/4\}$ then $\theta \eta = \{X/3, Y/f(4, 1)\}$.

- For all substitutions $\theta, \eta$ and $\gamma$ and an expression $E$
  i) $(E \theta) \eta \equiv E(\theta \eta)$
  ii) $(\theta \eta) \gamma = \theta(\eta \gamma)$.

- $\theta$ is more general than $\eta$ if for some $\gamma$ we have $\eta = \theta \gamma$.

- Example: $\theta = \{X/f(Y)\}$ more general than $\eta = \{X/f(h(G))\}$.
Unifiers

- If $A \theta \equiv B \theta$, then
  - $\theta$ is called a unifier of $A$ and $B$
  - $A$ and $B$ are unifiable

- A unifier $\theta$ of $A$ and $B$ is called a most general unifier (mgu) if it is more general than any other unifier of $A$ and $B$.

- If two atoms are unifiable then they have a most general unifier.

- $\theta$ is idempotent if $\theta \theta = \theta$.

- A unifier $\theta$ of $A$ and $B$ is relevant if all variables appearing either in $\text{domain}(\theta)$ or in $\text{range}(\theta)$, also appear in $A$ or $B$.

- If two atoms are unifiable then they have an mgu which is idempotent and relevant.

- An mgu is unique up to renaming.
Unification Algorithm

- Non-deterministically choose from the set of equations an equation of a form below and perform the associated action.
  1. \( f(s_1, \ldots, s_n) = f(t_1, \ldots, t_n) \rightarrow \text{replace by } s_1 = t_1, \ldots, s_n = t_n \)
  2. \( f(s_1, \ldots, s_n) = g(t_1, \ldots, t_m) \) where \( f \not\equiv g \rightarrow \text{halt with failure} \)
  3. \( X = X \rightarrow \text{delete the equation} \)
  4. \( t = X \) where \( t \) is not a variable \( \rightarrow \text{replace by the equation } X = t \)
  5. \( X = t \) where \( X \not\equiv t \) and \( X \) has another occurrence in the set of equations \( \rightarrow \)
     5.1 if \( X \) appears in \( t \) then halt with failure
     5.2 otherwise apply \( \{X/t\} \) to every other equation

- Consider the set of equations \( \{f(x) = f(f(z)), g(a, y) = g(a, x)\} \):
  - (1) produces \( \{x = f(z), g(a, y) = g(a, x)\} \)
  - then (1) yields \( \{x = f(z), a = a, y = x\} \)
  - (3) produces \( \{x = f(z), y = x\} \)
  - now only (5) can be applied, giving \( \{x = f(z), y = f(z)\} \)
  - No step can be applied, the algorithm successfully terminates.
Unification Algorithm revisited

- Let $A$ and $B$ be two formulas:
  1. $\theta = \epsilon$
  2. while $A\theta \neq B\theta$:
     2.1 find leftmost symbol in $A\theta$ s.t. the corresponding symbol in $B\theta$ is different
     2.2 let $t_A$ and $t_B$ be the terms in $A\theta$ and $B\theta$ starting with those symbols
        (a) if neither $t_A$ nor $t_B$ are variables or one is a variable occurring in the other → halt with failure
        (b) otherwise, let $t_A$ be a variable → the new $\theta$ is the result of $\theta\{t_A/t_B\}$
  3. end with $\theta$ being an m.g.u. of $A$ and $B$
Unification Algorithm revisited (Contd.)

- **Example:** \( A = p(X, X) \) \( B = p(f(A), f(B)) \)

<table>
<thead>
<tr>
<th>( \theta )</th>
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<th>( B\theta )</th>
<th>Element</th>
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<td>( p(f(A), f(B)) )</td>
<td>{( A/B }}</td>
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<tr>
<td>{( X/f(B), A/B }}</td>
<td>( p(f(B), f(B)) )</td>
<td>( p(f(B), f(B)) )</td>
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- **Example:** \( A = p(X, f(Y)) \) \( B = p(Z, X) \)

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Resolution with Variables

- It is a *formal system* with:
  - A first order language with the following formulas:
    * Clauses: without repetition, and without an order among their literals.
    * The empty clause \( \square \).
  - An empty set of axioms.
  - Two inference rules: *resolution* and *replacement*. 
Resolution with Variables (Contd.)

- **Resolution:**

\[
\begin{align*}
  r_1 &: A \lor F_1 \lor \cdots \lor F_n \\
  r_2 &: \neg B \lor G_1 \lor \cdots \lor G_m
\end{align*}
\]

\[
\frac{(F_1 \lor \cdots \lor F_n)\sigma \lor G_1 \lor \cdots \lor G_m)\theta}{\text{resolvent of } r_1 \text{ and } r_2}
\]

where

- $A$ and $B$ are unifiable with substitution $\theta$
- $\sigma$ is a renaming s.t. $(A \lor F_1 \lor \cdots \lor F_n)\sigma$ and $\neg B \lor G_1 \lor \cdots \lor G_m$ have no variables in common
- $\theta$ is the m.g.u. of $A\sigma$ and $B$

The resulting clause is called the *resolvent* of $r_1$ and $r_2$.

- **Replacement:** $A \lor B \lor F_1 \lor \cdots \lor F_n \Rightarrow (A \lor F_1 \lor \cdots \lor F_n)\theta$ where

  - $A$ and $B$ are unifiable atoms
  - $\theta$ is the m.g.u. of $A$ and $B$
Basic Properties

• Resolution is correct – i.e. all conclusions obtained using it are valid.

• There is no guarantee of directly deriving a given theorem.

• However, resolution (under certain assumptions) is refutation complete: if we have a set of clauses $K = [C_0, C_1, \ldots, C_n]$ and it is inconsistent then resolution will arrive at the empty clause $\square$ in a finite number of steps.

• Therefore, a valid theorem (or a question that has an answer) is guaranteed to be provable by refutation. To prove “p” given $K_0 = [C_0, C_1, \ldots, C_n]$:
  1. Negate it ($\neg p$).
  2. Construct $K = [\neg p, C_0, C_1, \ldots, C_n]$.
  3. Apply resolution steps repeatedly to K.

• Furthermore, we can obtain answers by composing the substitutions along a path that leads to $\square$ (very important for realizing Greene’s dream!).

• It is important to use a good method in applying the resolution steps – i.e. in building the resolution tree (or proof tree).

• Again, the main issue is to reduce the branching factor.
Proof Tree

- Given a set of clauses $K = \{C_0, C_1, \cdots, C_n\}$ the proof tree of $K$ is a tree s.t.:
  - the root is $C_0$
  - the branch from the root starts with the nodes labeled with $C_0, C_1, \cdots, C_n$
  - the descendents of $C_n$ are labeled by clauses obtained from the parent clauses using resolution
  - a derivation in $K$ is a branch of the proof tree of $K$

- The derivation $C_0 C_1 \cdots C_n F_0 \cdots F_m$ is denoted as $K, F_0 \cdots F_m$
Example: part of the proof tree for $K$, with:

$$K = \{ p, \neg p \lor q, \neg q \}$$

- $p \equiv C_0$
- $\neg p \lor q \equiv C_1$
- $\neg q \equiv C_2$

- $R(C_0,C_1) \equiv q$
- $\neg p \equiv R(C_1,C_2)$
- $\neg p \equiv R(C_1,C_2)$
- $R(C_0,C_1) \equiv q$
Characteristics of the Proof Tree

- It can be infinite:

\[ K = [ \neg p(X) \lor p(f(X)) ] \]

\[
\begin{align*}
p(e) & \equiv C0 \\
\neg p(X) \lor p(f(X)) & \equiv C1 \\
p(f(e)) & \quad \Theta = \{ X/e \} \\
p(f(f(e))) & \quad \Theta = \{ X/f(e) \} 
\end{align*}
\]

- Even if it is finite, it can be too large to be explored efficiently

- Aim: determine some criteria to limit the number of derivations and the way in which the tree is explored \( \Rightarrow \) strategy

- Any strategy based on this tree is correct: if \( \Box \) appears in a subtree of the proof tree of \( K \), then \( \Box \) can be derived from \( K \) and therefore \( K \) is unsatisfiable
General Strategies

- **Depth-first with backtracking**: First descendant to the left; if failure or $\square$ then backtrack

```
   1
  /  \
 2    7
 / \  /  \\
3   6  8  9
  |   |  |  |
4   Fail 8 9
  |   |  |  |
5   Fail  |  |
      |  |
      Fail
```

Fail Fail Fail
General Strategies (Contd.)

- **Breadth first**: all sons of all sibling nodes from left to right

```
  1
 / \
2   3
/    /
4   5 6   7
/    /   |
8    Fail 9 Fail
```
• **Iterative deepening**
  ◦ Advance depth-first for a time.
  ◦ After a certain depth, switch to another branch as in breadth-first.

• **Completeness issues / possible types of branches:**
  ◦ Success (always finite)
  ◦ Finite failure
  ◦ Infinite failure (provably infinite branches)
  ◦ Non-provably infinite branches
Linear Strategies

- Those which only explore linear derivations
- A derivation $K, F_0 \cdots F_m$ is linear if
  - $F_0$ is obtained by resolution or replacement using $C_0$
  - $F_i, i < 0$ is obtained by resolution or replacement using $F_{i-1}$
- Examples:

```
p \equiv C0
\neg p \lor q \equiv C0
\neg p \lor q \equiv C1
\neg q \equiv C0
\neg q \equiv C1
\neg q \equiv C2
q \equiv F0
q \equiv F0
\neg p \quad q
\neg p \quad q
```
Characteristics of these Strategies

1. If □ can be derived from $K$ by using resolution with variables, it can also be derived by linear resolution.

2. Let $K$ be $K' \cup \{C_0\}$ where $K'$ is a satisfiable set of clauses, i.e. □ cannot be derived from $K'$ by using resolution with variables. If □ can be derived from $K$ by using resolution with variables it can also be derived by linear resolution with root $C_0$.

- From (1), if the strategy is breadth first, it is complete.
- From (2), if we want to prove that $B$ is derived from $K'$ then we can apply linear resolution to $K = K' \cup \{\neg B\}$.
- Depth first with backtracking is not complete:

```
K = [ p(e), \neg p(X) \lor p(f(X)), \neg p(X) ]

C0

\neg p(X) \lor p(f(X))
C1

\neg p(X) \equiv C2

F0 \equiv p(f(e))

F1 \equiv p(f(f(e)))

\square

\square

\square
```
Input Strategies

- Those which only explore input derivations
- A derivation $K, F_0 \ldots F_m$ is input if
  - $F_0$ is obtained by resolution or replacement using $C_0$
  - $F_i, i < 0$ is obtained by resolution or replacement using at least a clause in $K$

Example:

$K = [-p \lor q, p \lor \neg r, r, q \lor \neg s, s \lor q]$

```
- p \lor q \quad \equiv \quad C_0
  p \lor \neg r \quad \equiv \quad C_1
  r \quad \equiv \quad C_2
  q \lor \neg s \quad \equiv \quad C_3
  s \lor q \quad \equiv \quad C_4

\neg q \lor \neg r \quad \leftarrow \quad C_1 (\& C_0)
```

Input + Linear

```
\neg q \quad \leftarrow \quad C_2
\neg s \quad \leftarrow \quad C_3
q \quad \leftarrow \quad C_4
\neg p \quad \leftarrow \quad C_0
\neg r \quad \leftarrow \quad C_1
\neg r \quad \leftarrow \quad C_2
```
Input Strategies

- In an input derivation, if $F_{i-1}$ does not appear in any derivation of a successor clause, it can be eliminated from the derivation without changing the result.
- If $F_{i-1}$ appears in the derivation of $F_j$, $j > 1$, $F_{i-1}$ can be allocated in position $j - 1$.
- As a result, we can limit ourselves to linear input derivations without losing any input derivable clause.
- Let $K$ be $K^r \cup \{C_0\}$ where $\Box$ is derived by using resolution with variables, $C_0$ is a negative Horn clause and all clauses in $K^r$ are positive Horn clauses. There is an input derivation with root $C_0$ finishing in $\Box$ and in which the replacement rule is not used (Hernschen 1974).
- A Horn clause is a clause in which at most one literal is positive:
  - it is positive if precisely one literal is positive
  - it is negative if all literals are negatives
- As a result, in those conditions, a breadth first input strategy is complete, and a depth first input strategy with backtracking is complete if the tree is finite.
Ordered Strategies

- We consider a new formal system in which:
  1. clauses are \textit{ordered} sets
  2. ordered resolution of two clauses
     \[ A = p_1 \lor \cdots \lor p_n \quad \text{and} \quad B = q_1 \lor \cdots \lor q_m \]
     where \( p_1 \) is a positive literal and \( q_1 \) is a negative literal is possible iff \( \neg p_1 \) and \( \sigma(q_1) \) are unifiable (\( \sigma \) is a renaming, s.t. \( p_1 \) and \( \sigma(q_1) \) have no variables in common)
  3. the resolvent of \( A \) and \( B \) is \( \theta(p_2 \lor \cdots \lor p_n \lor \sigma(q_2 \lor \cdots \lor q_m)) \) where \( \theta \) is an m.g.u of \( \neg p_1 \) and \( \sigma(q_1) \)

- Let \( K = K' \cup \{ C_0 \} \) be a set of clauses s.t. \( \square \) is derived by using resolution with variables, \( C_0 \) is a negative Horn clause and all clauses in \( K' \) are positive Horn clauses with the positive literal in the first place. There is a sorted input derivation with root \( C_0 \) arriving at \( \square \).

- In this context a sorted linear input with:
  - breadth first: is complete
  - depth first with backtracking: is complete if the tree is finite