Programming and Computational Logic
A Motivational Introduction

The following people have contributed to this course material:

Manuel Hermenegildo (editor), Technical University of Madrid, Spain and University of New Mexico, USA; Francisco Bueno, Manuel Carro, Pedro López, and Daniel Cabeza, Technical University of Madrid, Spain; María José García de la Banda, Monash University, Australia; David H. D. Warren, University of Bristol, U.K.; Ulrich Neumerkel, Technical University of Vienna, Austria; Michael Codish, Ben Gurion University, Israel
Course General Topic

Computational Logic

- logic
- algorithms
- lambda calculus
- logic and AI
- knowledge representation
- functional programming
- logic of programming
- constraints
- declarative programming

Logic of Computation
- program verification
- proving properties

Declarative Programming
- direct use of logic
- as a programming tool
The Program Correctness Problem

- Conventional models of using computers – not easy to determine correctness!
  - Has become a very important issue, not just in safety-critical apps.
  - Components with assured quality, being able to give a warranty, ...
  - Being able to run untrusted code, certificate carrying code, ...
A Simple Imperative Program

- Example:

```c
#include <stdio.h>
main() {
    int Number, Square;
    Number = 0;
    while(Number <= 5)
        { Square = Number * Number;
          printf("%d\n",Square);
          Number = Number + 1; } }
```

- Is it correct? With respect to what?

- A suitable formalism:
  - to provide *specifications* (describe problems), and
  - to reason about the *correctness of programs* (their *implementation*).

is needed.
“Compute the squares of the natural numbers which are less or equal than 5.”

Ideal at first sight, but:

- verbose
- vague
- ambiguous
- needs context (assumed information)
- ...

Philosophers and Mathematicians already pointed this out a long time ago...
Logic

- A means of clarifying / formalizing the human thought process
- Logic for example tells us that (classical logic)
  Aristotle likes cookies, and
  Plato is a friend of anyone who likes cookies
  imply that
  Plato is a friend of Aristotle

- Symbolic logic:
  A shorthand for classical logic – plus many useful results:
  \( a_1 : \text{likes}(\text{aristotle}, \text{cookies}) \)
  \( a_2 : \forall X \text{ likes}(X, \text{cookies}) \rightarrow \text{friend}(\text{plato}, X) \)
  \( t_1 : \text{friend}(\text{plato}, \text{aristotle}) \)
  \( T[a_1, a_2] \vdash t_1 \)

- But, can logic be used:
  - To represent the problem (specifications)?
  - Even perhaps to solve the problem?
For expressing specifications and reasoning about the correctness of programs we need:

- Specification languages (assertions), modeling, ...
- Program semantics (models, axiomatic, fixpoint, ...).
- Proofs: program verification (and debugging, equivalence, ...).
Generating Squares: A Specification (I)

Numbers—we will use “Peano” representation for simplicity:

\[
0 \rightarrow 0 \\
1 \rightarrow s(0) \\
2 \rightarrow s(s(0)) \\
3 \rightarrow s(s(s(0))) \\
\ldots
\]

• Defining the natural numbers:

\[\text{nat}(0) \land \text{nat}(s(0)) \land \text{nat}(s(s(0))) \land \ldots\]

• A better solution:

\[\text{nat}(0) \land \forall X \left(\text{nat}(X) \rightarrow \text{nat}(s(X))\right)\]

• Order on the naturals:

\[
\forall X \left(\text{le}(0, X)\right) \land \\
\forall X \forall Y \left(\text{le}(X, Y) \rightarrow \text{le}(s(X), s(Y))\right)
\]

• Addition of naturals:

\[
\forall X \left(\text{nat}(X) \rightarrow \text{add}(0, X, X)\right) \land \\
\forall X \forall Y \forall Z \left(\text{add}(X, Y, Z) \rightarrow \text{add}(s(X), Y, s(Z))\right)
\]
Generating Squares: A Specification (II)

- Multiplication of naturals:
  \[ \forall X \ (\text{nat}(X) \to \text{mult}(0, X, 0)) \land \]
  \[ \forall X \forall Y \forall Z \forall W \ (\text{mult}(X, Y, W) \land \text{add}(W, Y, Z) \to \text{mult}(s(X), Y, Z)) \]

- Squares of the naturals:
  \[ \forall X \forall Y \ (\text{nat}(X) \land \text{nat}(Y) \land \text{mult}(X, X, Y) \to \text{nat\_square}(X, Y)) \]

We can now write a specification of the (imperative) program, i.e., conditions that we want the program to meet:

- **Precondition:**
  empty.

- **Postcondition:**
  \[ \forall X (\text{output}(X) \leftarrow (\exists Y \ \text{nat}(Y) \land \text{le}(Y, s(s(s(s(0))))) \land \text{nat\_square}(Y, X))) \]
For expressing specifications and reasoning about the correctness of programs we need:

- Specification languages (assertions), modeling, ...
- Program semantics (models, axiomatic, fixpoint, ...).
- Proofs: program verification (and debugging, equivalence, ...).
Semantic Tasks

- Semantics:
  - A semantics associates a meaning (a mathematical object) to a program or program sentence.

- Semantic tasks:
  - Verification: proving that a program meets its specification.
  - Static debugging: finding where a program does not meet specifications.
  - Program equivalence: proving that two programs have the same semantics.
  - etc.
Styles of Semantics

- **Operational:**
  The meaning of program sentences is defined in terms of the steps (transformations from state to state) that computations may take during execution (derivations). Proofs by induction on derivations.

- **Axiomatic:**
  The meaning of program sentences is defined indirectly in terms some axioms and rules of some logic of program properties.

- **Denotational (fixpoint):**
  The meaning of program sentences is given abstractly as elements of some suitable mathematical structure (domain).

- **Model (declarative) semantics:**
  The meaning of programs is given as a minimal model (“logical meaning”) of the logic that the program is written in.
Alternative Use of Logic?

- So, logic allows us to *represent problems* (program specifications).

  But, it would be interesting to also improve:

  i.e., the process of implementing solutions to problems.

- The importance of Programming Languages (and tools).

- Interesting question: can logic help here too?
• Assuming the existence of a *mechanical proof method* (deduction procedure) 
  *a new view of problem solving and computing is possible* [Greene]:
  ◊ program once and for all the deduction procedure in the computer,
  ◊ find a suitable *representation* for the problem (i.e., the *specification*),
  ◊ then, to obtain solutions, ask questions and let deduction procedure do rest:

- No correctness proofs needed!
### Computing With Our Previous Description / Specification

<table>
<thead>
<tr>
<th>Query</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>nat((s(0))) ?</td>
<td>(yes)</td>
</tr>
<tr>
<td>(\exists X \ add(s(0), s(s(0)), X) ?)</td>
<td>(X = s(s(s(0))))</td>
</tr>
<tr>
<td>(\exists X \ add(s(0), X, s(s(s(0)))) ?)</td>
<td>(X = s(s(0)))</td>
</tr>
<tr>
<td>(\exists X \ nat(X) ?)</td>
<td>(X = 0 \lor X = s(0) \lor X = s(s(0)) \lor \ldots)</td>
</tr>
<tr>
<td>(\exists X \exists Y \ add(X, Y, s(0)) ?)</td>
<td>((X = 0 \land Y = s(0)) \lor (X = s(0) \land Y = 0))</td>
</tr>
<tr>
<td>(\exists X \ nat\text{-}square((s(s(0))), (X)) ?)</td>
<td>(X = s(s(s(0)))))</td>
</tr>
<tr>
<td>(\exists X \ nat\text{-}square((X, s(s(s(s(0)))))) ?)</td>
<td>(X = s(s(0)))</td>
</tr>
<tr>
<td>(\exists X \exists Y \ nat\text{-}square((X, Y)) ?)</td>
<td>((X = 0 \land Y = 0) \lor (X = s(0) \land Y = s(0)) \lor (X = s(s(0)) \land Y = s(s(s(s(0)))))) \lor \ldots)</td>
</tr>
<tr>
<td>(\exists X \ output(X) ?)</td>
<td>(X = 0 \lor X = s(0) \lor X = s(s(s(s(0)))) \lor X = s^9(0) \lor X = s^{16}(0) \lor X = s^{25}(0))</td>
</tr>
</tbody>
</table>
Which Logic?

- We have already argued the convenience of representing the problem in logic, but
  - which logic?
    - propositional
    - predicate calculus (first order)
    - higher-order logics
    - modal logics
    - λ-calculus, ...
  - which reasoning procedure?
    - natural deduction, classical methods
    - resolution
    - Prawitz/Bibel, tableaux
    - bottom-up fixpoint
    - rewriting
    - narrowing, ...
Issues

- We try to maximize expressive power.
- But one of the main issues is whether we have an **effective** reasoning procedure.
- It is important to understand the underlying properties and the theoretical limits!
- Example: propositions vs. first-order formulas.
  - **Propositional logic:**
    
    - "spot is a dog" \( p \)
    - "dogs have tail" \( q \)
    
    but how can we conclude that Spot has a tail?
  - **Predicate logic** extends the expressive power of propositional logic:
    
    \[
    \text{dog}(\text{spot}) \\
    \forall X \text{dog}(X) \rightarrow \text{has\_tail}(X)
    \]
    
    now, using deduction we can conclude:
    
    \[
    \text{has\_tail}(\text{spot})
    \]
Comparison of Logics (I)

- Propositional logic:
  
  "spot is a dog" $p$
  + decidability/completeness
  - limited expressive power
  + practical deduction mechanism

  $\rightarrow$ circuit design, "answer set" programming, ...

- Predicate logic: (first order)
  
  "spot is a dog" $\text{dog}(\text{spot})$
  +/- decidability/completeness
  +/- good expressive power
  + practical deduction mechanism (e.g., SLD-resolution)

  $\rightarrow$ classical logic programming!
Comparison of Logics (II)

- Higher-order predicate logic:
  
  “There is a relationship for spot” \( \lambda \text{(spot)} \)
  
  - decidability/completeness
  
  + good expressive power
  
  – practical deduction mechanism

  But interesting subsets \( \rightarrow \) HO logic programming, functional-logic programming, ...

- Other logics: decidability? Expressive power? Practical deduction mechanism?
  
  Often (very useful) variants of previous ones:
  
    ◦ Predicate logic + constraints (in place of unification)
      
        \( \rightarrow \) constraint programming!
    
    ◦ Propositional temporal logic, etc.

- Interesting case: \( \lambda \)-calculus
  
  + similar to predicate logic in results, allows higher order
  
  - does not support predicates (relations), only functions

  \( \rightarrow \) functional programming!
Generating squares by SLD-Resolution – Logic Programming (I)

• We code the problem as definite (Horn) clauses:
  
  \[ \text{nat}(0) \]
  \[ \neg \text{nat}(X) \lor \text{nat}(s(X)) \]
  \[ \neg \text{nat}(X) \lor \text{add}(0, X, X) \]
  \[ \neg \text{add}(X, Y, Z) \lor \text{add}(s(X), Y, s(Z)) \]
  \[ \neg \text{nat}(X) \lor \text{mult}(0, X, 0) \]
  \[ \neg \text{mult}(X, Y, W) \lor \neg \text{add}(W, Y, Z) \lor \text{mult}(s(X), Y, Z) \]
  \[ \neg \text{nat}(X) \lor \neg \text{nat}(Y) \lor \neg \text{mult}(X, X, Y) \lor \text{nat\_square}(X, Y) \]

• \textbf{Query:} \quad \text{nat}(s(0)) \quad ?

• In order to refute: \quad \neg \text{nat}(s(0))

• \textbf{Resolution:}
  
  \[ \neg \text{nat}(s(0)) \text{ with } \neg \text{nat}(X) \lor \text{nat}(s(X)) \text{ gives } \neg \text{nat}(0) \]
  
  \[ \neg \text{nat}(0) \text{ with } \text{nat}(0) \text{ gives } \square \]

• \textbf{Answer:} \quad (yes)
Generating squares by SLD-Resolution – Logic Programming (II)

nat(0)
¬nat(X) ∨ nat(s(X))
¬nat(X) ∨ add(0, X, X))
¬add(X, Y, Z) ∨ add(s(X), Y, s(Z))
¬nat(X) ∨ mult(0, X, 0)
¬mult(X, Y, W) ∨ ¬add(W, Y, Z) ∨ mult(s(X), Y, Z)
¬nat(X) ∨ ¬nat(Y) ∨ ¬mult(X, X, Y) ∨ nat_square(X, Y)

• **Query:** ∃X∃Y add(X, Y, s(0))  ?

• In order to refute:  ¬add(X, Y, s(0))

• Resolution:
  ¬add(X, Y, s(0)) with ¬nat(X) ∨ add(0, X, X)) gives ¬nat(s(0))
  ¬nat(s(0)) solved as before

• Answer:  X = 0, Y = s(0)

• Alternative:
  ¬add(X, Y, s(0)) with ¬add(X, Y, Z) ∨ add(s(X), Y, s(Z)) gives ¬add(X, Y, 0)
Generating Squares in a Practical Logic Programming System (I)

:- module(_,_,['bf/af']).

nat(0) <- .
nat(s(X)) <- nat(X).

le(0,_X) <- .
le(s(X),s(Y)) <- le(X,Y).

add(0,Y,Y) <- nat(Y).
add(s(X),Y,s(Z)) <- add(X,Y,Z).

mult(0,Y,0) <- nat(Y).
mult(s(X),Y,Z) <- add(W,Y,Z), mult(X,Y,W).

nat_square(X,Y) <- nat(X), nat(Y), mult(X,X,Y).

output(X) <- nat(Y), le(Y,s(s(s(s(s(0)))))), nat_square(Y,X).
Generating Squares in a Practical Logic Programming System (II)

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</tr>
<tr>
<td><code>?- nat(X).</code></td>
<td><code>X = 0 ; X = s(0) ; X = s(s(0)) ; ...</code></td>
</tr>
<tr>
<td><code>?- add(X, Y, s(0)).</code></td>
<td><code>(X = 0 , Y=s(0)) ; (X = s(0) , Y = 0)</code></td>
</tr>
<tr>
<td><code>?- nat_square(s(0), X).</code></td>
<td><code>X = s(s(s(0))))</code></td>
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