The following people have contributed to this course material: 

Manuel Hermenegildo (editor), Technical University of Madrid, Spain and University of New Mexico, USA; Francisco Bueno, Manuel Carro, Pedro López, and Daniel Cabeza, Technical University of Madrid, Spain; María José García de la Banda, Monash University, Australia; David H. D. Warren, University of Bristol, U.K.; Ulrich Neumerkel, Technical University of Vienna, Austria; Michael Codish, Ben Gurion University, Israel
Course General Topic

Computational Logic

- logic
- algorithms
- lambda calculus
- logic and AI
- knowledge representation
- functional programming
- logic programming
- constraints
- logic of programming
- declarative programming

Logic of Computation
- program verification
- proving properties

Declarative Programming
- direct use of logic
- as a programming tool
The Program Correctness Problem

- Conventional models of using computers – not easy to determine correctness!
  - Has become a very important issue, not just in safety-critical apps.
  - Components with assured quality, being able to give a warranty, ...
  - Being able to run untrusted code, certificate carrying code, ...
A Simple Imperative Program

- Example:

```c
#include <stdio.h>
main() {
    int Number, Square;
    Number = 0;
    while(Number <= 5)
    {
        Square = Number * Number;
        printf("%d\n",Square);
        Number = Number + 1;
    }
}
```

- Is it correct? With respect to what?

- A suitable formalism:
  - to provide *specifications* (describe problems), and
  - to reason about the *correctness of programs* (their *implementation*).

is needed.
Natural Language

“Compute the squares of the natural numbers which are less or equal than 5.”

Ideal at first sight, but:

- verbose
- vague
- ambiguous
- needs context (assumed information)
- ...

Philosophers and Mathematicians already pointed this out a long time ago...
Logic

- A means of clarifying / formalizing the human thought process

- Logic for example tells us that (classical logic)
  *Aristotle likes cookies, and
  *Plato is a friend of anyone who likes cookies*
  imply that
  *Plato is a friend of Aristotle*

- Symbolic logic:
  A shorthand for classical logic – plus many useful results:
  
  \[ a_1 : \text{likes}(\text{aristotle, cookies}) \]
  
  \[ a_2 : \forall X \text{ likes}(X, \text{cookies}) \rightarrow \text{friend}(\text{plato, X}) \]
  
  \[ t_1 : \text{friend}(\text{plato, aristotle}) \]
  
  \[ T[a_1, a_2] \vdash t_1 \]

- But, can logic be used:
  
  - To represent the problem (specifications)?
  - *Even perhaps to solve the problem?*
For expressing specifications and reasoning about the correctness of programs we need:

- Specification languages (assertions), modeling, ...
- Program semantics (models, axiomatic, fixpoint, ...).
- Proofs: program verification (and debugging, equivalence, ...).
Generating Squares: A Specification (I)

Numbers—we will use “Peano” representation for simplicity:

\[0 \rightarrow 0 \quad 1 \rightarrow s(0) \quad 2 \rightarrow s(s(0)) \quad 3 \rightarrow s(s(s(0))) \ldots\]

- Defining the natural numbers:
  \[nat(0) \land nat(s(0)) \land nat(s(s(0))) \land \ldots\]

- A better solution:
  \[nat(0) \land \forall X (nat(X) \rightarrow nat(s(X)))\]

- Order on the naturals:
  \[\forall X (le(0, X)) \land \forall X \forall Y (le(X, Y) \rightarrow le(s(X), s(Y)))\]

- Addition of naturals:
  \[\forall X (nat(X) \rightarrow add(0, X, X)) \land \forall X \forall Y \forall Z (add(X, Y, Z) \rightarrow add(s(X), Y, s(Z))))\]
Generating Squares: A Specification (II)

- Multiplication of naturals:
  \[ \forall X \ (\text{nat}(X) \rightarrow \text{mult}(0, X, 0)) \land \]
  \[ \forall X \forall Y \forall Z \forall W \ (\text{mult}(X, Y, W) \land \text{add}(W, Y, Z) \rightarrow \text{mult}(s(X), Y, Z)) \]

- Squares of the naturals:
  \[ \forall X \forall Y \ (\text{nat}(X) \land \text{nat}(Y) \land \text{mult}(X, X, Y) \rightarrow \text{nat_square}(X, Y)) \]

We can now write a specification of the (imperative) program, i.e., conditions that we want the program to meet:

- **Precondition:**
  empty.

- **Postcondition:**
  \[ \forall X (\text{output}(X) \leftarrow (\exists Y \ \text{nat}(Y) \land \text{le}(Y, s(s(s(s(0))))) \land \text{nat_square}(Y, X))) \]
For expressing specifications and reasoning about the correctness of programs we need:

- Specification languages (assertions), modeling, ...
- Program semantics (models, axiomatic, fixpoint, ...).
- Proofs: program verification (and debugging, equivalence, ...).
Semantic Tasks

- Semantics:
  - A *semantics* associates a meaning (a mathematical object) to a program or program sentence.

- Semantic tasks:
  - Verification: proving that a program meets its specification.
  - Static debugging: finding where a program does not meet specifications.
  - Program equivalence: proving that two programs have the same semantics.
  - etc.
**Styles of Semantics**

- **Operational:**
  The meaning of program sentences is defined in terms of the steps (transformations from state to state) that computations may take during execution (derivations). Proofs by induction on derivations.

- **Axiomatic:**
  The meaning of program sentences is defined indirectly in terms some axioms and rules of some logic of program properties.

- **Denotational (fixpoint):**
  The meaning of program sentences is given abstractly as elements of some suitable mathematical structure (domain).

- **Model (declarative) semantics:**
  The meaning of programs is given as a minimal model ("logical meaning") of the logic that the program is written in.
Alternative Use of Logic?

- So, logic allows us to *represent problems* (program specifications).

But, it would be interesting to also improve:

- i.e., the process of implementing solutions to problems.
- The importance of Programming Languages (and tools).
- Interesting question: can logic help here too?
From Representation/Specification to Computation

- Assuming the existence of a *mechanical proof method* (deduction procedure) a *new view of problem solving and computing is possible* [Green]:
  - program once and for all the deduction procedure in the computer,
  - find a suitable *representation* for the problem (i.e., the *specification*),
  - then, to obtain solutions, ask questions and let deduction procedure do rest:

- No correctness proofs needed!
### Computing With Our Previous Description / Specification

<table>
<thead>
<tr>
<th>Query</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>$nat(s(0))$ ?</td>
<td>(yes)</td>
</tr>
<tr>
<td>$\exists X \ add(s(0), s(s(0)), X) ?$</td>
<td>$X = s(s(s(0)))$</td>
</tr>
<tr>
<td>$\exists X \ add(s(0), X, s(s(s(0)))) ?$</td>
<td>$X = s(s(0))$</td>
</tr>
<tr>
<td>$\exists X \ nat(X) ?$</td>
<td>$X = 0 \lor X = s(0) \lor X = s(s(0)) \lor \ldots$</td>
</tr>
<tr>
<td>$\exists X\exists Y \ add(X, Y, s(0)) ?$</td>
<td>$(X = 0 \land Y = s(0)) \lor (X = s(0) \land Y = 0)$</td>
</tr>
<tr>
<td>$\exists X \ nat_square(s(s(0)), X) ?$</td>
<td>$X = s(s(s(0))))$</td>
</tr>
<tr>
<td>$\exists X \ nat_square(X, s(s(s(0)))) ?$</td>
<td>$X = s(s(0))$</td>
</tr>
<tr>
<td>$\exists X\exists Y \ nat_square(X, Y) ?$</td>
<td>$(X = 0 \land Y = 0) \lor (X = s(0) \land Y = s(0)) \lor (X = s(s(0))) \land Y = s(s(s(0)))) \lor \ldots$</td>
</tr>
<tr>
<td>$\exists X output(X) ?$</td>
<td>$X = 0 \lor X = s(0) \lor X = s(s(s(s(0)))) \lor X = s^9(0) \lor X = s^{16}(0) \lor X = s^{25}(0)$</td>
</tr>
</tbody>
</table>
Which Logic?

- We have already argued the convenience of representing the problem in logic, but
  - which logic?
    - propositional
    - predicate calculus (first order)
    - higher-order logics
    - modal logics
    - $\lambda$-calculus, ...
  - which reasoning procedure?
    - natural deduction, classical methods
    - resolution
    - Prawitz/Bibel, tableaux
    - bottom-up fixpoint
    - rewriting
    - narrowing, ...

• We try to maximize expressive power.
• But one of the main issues is whether we have an **effective** reasoning procedure.
• It is important to understand the underlying properties and the theoretical limits!
• Example: propositions vs. first-order formulas.
  ◦ Propositional logic:
    
    “spot is a dog” \( p \)
    
    “dogs have tail” \( q \)
    
    but how can we conclude that Spot has a tail?

  ◦ Predicate logic extends the expressive power of propositional logic:

\[
\text{dog}(\text{spot}) \\
\forall X \text{dog}(X) \rightarrow \text{has\_tail}(X)
\]

now, using deduction we can conclude:

\[
\text{has\_tail}(\text{spot})
\]
Comparison of Logics (I)

- Propositional logic:
  “spot is a dog” \( p \)
  + decidability/completeness
  - limited expressive power
  + practical deduction mechanism

  → circuit design, “answer set” programming, ...

- Predicate logic: (first order)
  “spot is a dog” \( dog(spot) \)
  +/- decidability/completeness
  +/- good expressive power
  + practical deduction mechanism (e.g., SLD-resolution)

  → classical logic programming!
Comparison of Logics (II)

- Higher-order predicate logic:
  “There is a relationship for spot” \( X(spot) \)
  - decidability/completeness
  + good expressive power
  – practical deduction mechanism

  But interesting subsets \( \rightarrow \) HO logic programming, functional-logic programming, ...

- Other logics: decidability? Expressive power? Practical deduction mechanism?
  Often (very useful) variants of previous ones:
  ◦ Predicate logic + constraints (in place of unification)
    \( \rightarrow \) constraint programming!
  ◦ Propositional temporal logic, etc.

- Interesting case: \( \lambda \)-calculus
  + similar to predicate logic in results, allows higher order
  - does not support predicates (relations), only functions

  \( \rightarrow \) functional programming!
Generating squares by SLD-Resolution – Logic Programming (I)

- We code the problem as definite (Horn) clauses:
  
  \[
  \begin{align*}
  & \text{nat}(0) \\
  & \neg \text{nat}(X) \lor \text{nat}(s(X)) \\
  & \neg \text{nat}(X) \lor \text{add}(0, X, X) \\
  & \neg \text{add}(X, Y, Z) \lor \text{add}(s(X), Y, s(Z)) \\
  & \neg \text{nat}(X) \lor \text{mult}(0, X, 0) \\
  & \neg \text{mult}(X, Y, W) \lor \neg \text{add}(W, Y, Z) \lor \text{mult}(s(X), Y, Z) \\
  & \neg \text{nat}(X) \lor \neg \text{nat}(Y) \lor \neg \text{mult}(X, X, Y) \lor \text{nat\_square}(X, Y)
  \end{align*}
  \]

- **Query:** \( \text{nat}(s(0)) \) ?

- In order to refute: \( \neg \text{nat}(s(0)) \)

- Resolution:
  
  \[
  \begin{align*}
  & \neg \text{nat}(s(0)) \text{ with } \neg \text{nat}(X) \lor \text{nat}(s(X)) \text{ gives } \neg \text{nat}(0) \\
  & \neg \text{nat}(0) \text{ with } \text{nat}(0) \text{ gives } \Box
  \end{align*}
  \]

- Answer: \( \text{(yes)} \)
Generating squares by SLD-Resolution – Logic Programming (II)

\[\begin{align*}
nat(0) \\
\neg nat(X) & \lor nat(s(X)) \\
\neg nat(X) & \lor add(0, X, X)) \\
\neg add(X, Y, Z) & \lor add(s(X), Y, s(Z)) \\
\neg nat(X) & \lor mult(0, X, 0) \\
\neg mult(X, Y, W) & \lor \neg add(W, Y, Z) \lor mult(s(X), Y, Z) \\
\neg nat(X) & \lor \neg nat(Y) \lor \neg mult(X, X, Y) \lor \neg nat\_square(X, Y)
\end{align*}\]

- **Query:** \(\exists X \exists Y \ add(X, Y, s(0)) \ ?\)
- **In order to refute:** \(\neg add(X, Y, s(0))\)
- **Resolution:**
  \(\neg add(X, Y, s(0)) \text{ with } \neg nat(X) \lor add(0, X, X)) \text{ gives } \neg nat(s(0))\)
  \(\neg nat(s(0)) \text{ solved as before}\)
- **Answer:** \(X = 0, Y = s(0)\)
- **Alternative:**
  \(\neg add(X, Y, s(0)) \text{ with } \neg add(X, Y, Z) \lor add(s(X), Y, s(Z)) \text{ gives } \neg add(X, Y, 0)\)
Generating Squares in a Practical Logic Programming System (I)

:- module(_,_,['bf/af']).

nat(0) <- .
nat(s(X)) <- nat(X).

le(0,X) <- .
le(s(X),s(Y)) <- le(X,Y).

add(0,Y,Y) <- nat(Y).
add(s(X),Y,s(Z)) <- add(X,Y,Z).

mult(0,Y,0) <- nat(Y).
mult(s(X),Y,Z) <- add(W,Y,Z), mult(X,Y,W).

nat_square(X,Y) <- nat(X), nat(Y), mult(X,X,Y).

output(X) <- nat(Y), le(Y,s(s(s(s(s(0)))))), nat_square(Y,X).
### Generating Squares in a Practical Logic Programming System (II)

<table>
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<tr>
<td><code>?- nat(s(0)).</code></td>
<td>yes</td>
</tr>
<tr>
<td><code>?- add(s(0),s(s(0)),X).</code></td>
<td><code>X = s(s(s(0)))</code></td>
</tr>
<tr>
<td><code>?- add(s(0),X,s(s(s(0)))).</code></td>
<td><code>X = s(s(0))</code></td>
</tr>
<tr>
<td><code>?- nat(X).</code></td>
<td><code>X = 0 ; X = s(0) ; X = s(s(0)) ; ...</code></td>
</tr>
<tr>
<td><code>?- add(X,Y,s(0)).</code></td>
<td><code>(X = 0 , Y=s(0)) ; (X = s(0) , Y = 0)</code></td>
</tr>
<tr>
<td><code>?- nat_square(s(s(0)), X).</code></td>
<td><code>X = s(s(s(s(0))))</code></td>
</tr>
<tr>
<td><code>?- nat_square(X,s(s(s(s(0))))).</code></td>
<td><code>X = s(s(0))</code></td>
</tr>
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<td><code>?- nat_square(X,Y).</code></td>
<td><code>(X = 0 , Y=0) ; (X = s(0) , Y=s(0)) ; (X = s(s(0)) , Y=s(s(s(s(0)))))) ; ...</code></td>
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<td><code>?- output(X).</code></td>
<td><code>X = 0 ; X = s(0) ; X = s(s(s(s(0)))) ; ...</code></td>
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