Programming and Computational Logic
A Motivational Introduction

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Course General Topic

Computational Logic

- logic
- algorithms
- verification
- lambda calculus
- logic programming
- logic of programming
- functional programming
- knowledge representation
- constraints
- declarative programming

Logic of Computation
- program verification
- proving properties

Declarative Programming
- direct use of logic
- as a programming tool
• Conventional models of using computers – not easy to determine correctness!
  ◊ Has become a very important issue, not just in safety-critical apps.
  ◊ Components with assured quality, being able to give a warranty, ... 
  ◊ Being able to run untrusted code, certificate carrying code, ...
A Simple Imperative Program

- Example:

```c
#include <stdio.h>
main() {
    int Number, Square;
    Number = 0;
    while(Number <= 5)
    {
        Square = Number * Number;
        printf("%d\n",Square);
        Number = Number + 1; } }
```

- Is it correct? With respect to what?

- A suitable formalism:
  - to provide *specifications* (describe problems), and
  - to reason about the *correctness of programs* (their *implementation*).

is needed.
Natural Language

“Compute the squares of the natural numbers which are less or equal than 5.”

Ideal at first sight, but:

- verbose
- vague
- ambiguous
- needs context (assumed information)
- ...

Philosophers and Mathematicians already pointed this out a long time ago...
Logic

• A means of clarifying / formalizing the human thought process

• Logic for example tells us that (classical logic)
  Aristotle likes cookies, and
  Plato is a friend of anyone who likes cookies
  imply that
  Plato is a friend of Aristotle

• Symbolic logic:
  A shorthand for classical logic – plus many useful results:
  \( a_1 : \text{likes}(\text{aristotle, cookies}) \)
  \( a_2 : \forall X \text{ likes}(X, \text{cookies}) \rightarrow \text{friend}(\text{plato, } X) \)
  \( t_1 : \text{friend}(\text{plato, aristotle}) \)
  \( T[a_1, a_2] \vdash t_1 \)

• But, can logic be used:
  ◇ To represent the problem (specifications)?
  ◇ *Even perhaps to solve the problem?*
For expressing specifications and reasoning about the correctness of programs we need:

- Specification languages (assertions), modeling, ...
- Program semantics (models, axiomatic, fixpoint, ...).
- Proofs: program verification (and debugging, equivalence, ...).
Generating Squares: A Specification (I)

Numbers—we will use “Peano” representation for simplicity:
0 → 0 1 → s(0) 2 → s(s(0)) 3 → s(s(s(0))) ... 

- Defining the natural numbers:
  \[ \text{nat}(0) \land \text{nat}(s(0)) \land \text{nat}(s(s(0))) \land \ldots \]

- A better solution:
  \[ \text{nat}(0) \land \forall X (\text{nat}(X) \rightarrow \text{nat}(s(X))) \]

- Order on the naturals:
  \[
  \forall X \ (\text{le}(0, X)) \land \\
  \forall X \forall Y \ (\text{le}(X, Y) \rightarrow \text{le}(s(X), s(Y)))
  \]

- Addition of naturals:
  \[
  \forall X \ (\text{nat}(X) \rightarrow \text{add}(0, X, X)) \land \\
  \forall X \forall Y \forall Z \ (\text{add}(X, Y, Z) \rightarrow \text{add}(s(X), Y, s(Z)))
  \]
Generating Squares: A Specification (II)

- **Multiplication of naturals:**
  \[
  \forall X \ (\text{nat}(X) \rightarrow \text{mult}(0, X, 0)) \land \\
  \forall X \forall Y \forall Z \forall W \ (\text{mult}(X, Y, W) \land \text{add}(W, Y, Z) \rightarrow \text{mult}(s(X), Y, Z))
  \]

- **Squares of the naturals:**
  \[
  \forall X \forall Y \ (\text{nat}(X) \land \text{nat}(Y) \land \text{mult}(X, X, Y) \rightarrow \text{nat_square}(X, Y))
  \]

We can now write a *specification* of the (imperative) program, i.e., conditions that we want the program to meet:

- **Precondition:**
  empty.

- **Postcondition:**
  \[
  \forall X (\text{output}(X) \leftarrow (\exists Y \ \text{nat}(Y) \land \text{le}(Y, s(s(s(s(0)))))) \land \text{nat_square}(Y, X))
  \]
For expressing specifications and reasoning about the correctness of programs we need:

- Specification languages (assertions), modeling, ...
- Program semantics (models, axiomatic, fixpoint, ...).
- Proofs: program verification (and debugging, equivalence, ...).
Semantic Tasks

- Semantics:
  - A semantics associates a meaning (a mathematical object) to a program or program sentence.

- Semantic tasks:
  - Verification: proving that a program meets its specification.
  - Static debugging: finding where a program does not meet specifications.
  - Program equivalence: proving that two programs have the same semantics.
  - etc.
Styles of Semantics

- **Operational:**
  The meaning of program sentences is defined in terms of the steps (transformations from state to state) that computations may take during execution (derivations). Proofs by induction on derivations.

- **Axiomatic:**
  The meaning of program sentences is defined indirectly in terms some axioms and rules of some logic of program properties.

- **Denotational (fixpoint):**
  The meaning of program sentences is given abstractly as elements of some suitable mathematical structure (domain).

- **Model (declarative) semantics:**
  The meaning of programs is given as a minimal model (“logical meaning”) of the logic that the program is written in.
Alternative Use of Logic?

- So, logic allows us to *represent problems* (program specifications).

But, it would be interesting to also improve:

- i.e., the process of implementing solutions to problems.
- The importance of Programming Languages (and tools).
- Interesting question: can logic help here too?
Assuming the existence of a *mechanical proof method* (deduction procedure), a new view of problem solving and computing is possible [Green]:

- program once and for all the deduction procedure in the computer,
- find a suitable *representation* for the problem (i.e., the *specification*),
- then, to obtain solutions, ask questions and let deduction procedure do rest:

- No correctness proofs needed!
### Computing With Our Previous Description / Specification

<table>
<thead>
<tr>
<th>Query</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>nat(s(0))</code> ?</td>
<td><code>(yes)</code></td>
</tr>
<tr>
<td><code>∃X add(s(0), s(s(0)), X)</code> ?</td>
<td><code>X = s(s(s(0)))</code></td>
</tr>
<tr>
<td><code>∃X add(s(0), X, s(s(s(0))))</code> ?</td>
<td><code>X = s(s(0))</code></td>
</tr>
<tr>
<td><code>∃X nat(X)</code> ?</td>
<td><code>X = 0 \lor X = s(0) \lor X = s(s(0)) \lor \ldots</code></td>
</tr>
<tr>
<td><code>∃X∃Y add(X, Y, s(0))</code> ?</td>
<td><code>(X = 0 \land Y = s(0)) \lor (X = s(0) \land Y = 0)</code></td>
</tr>
<tr>
<td><code>∃X nat_square(s(s(0)), X)</code> ?</td>
<td><code>X = s(s(s(s(0))))</code></td>
</tr>
<tr>
<td><code>∃X nat_square(X, s(s(s(s(0)))))</code> ?</td>
<td><code>X = s(s(0))</code></td>
</tr>
<tr>
<td><code>∃X∃Y nat_square(X, Y)</code> ?</td>
<td><code>(X = 0 \land Y = 0) \lor (X = s(0) \land Y = s(0)) \lor (X = s(s(0)) \land Y = s(s(s(s(0)))) \lor \ldots</code></td>
</tr>
<tr>
<td><code>∃X output(X)</code> ?</td>
<td><code>X = 0 \lor X = s(0) \lor X = s(s(s(s(s(0))))) \lor X = s^9(0) \lor X = s^{16}(0) \lor X = s^{25}(0)</code></td>
</tr>
</tbody>
</table>
Which Logic?

- We have already argued the convenience of representing the problem in logic, but
  - which logic?
    - propositional
    - predicate calculus (first order)
    - higher-order logics
    - modal logics
    - $\lambda$-calculus, ...
  - which reasoning procedure?
    - natural deduction, classical methods
    - resolution
    - Prawitz/Bibel, tableaux
    - bottom-up fixpoint
    - rewriting
    - narrowing, ...
Issues

- We try to maximize expressive power.
- But one of the main issues is whether we have an effective reasoning procedure.
- It is important to understand the underlying properties and the theoretical limits!
- Example: propositions vs. first-order formulas.
  - Propositional logic:
    - “spot is a dog” \( p \)
    - “dogs have tail” \( q \)
    - but how can we conclude that Spot has a tail?
  - Predicate logic extends the expressive power of propositional logic:
    - \( \text{dog}(\text{spot}) \)
    - \( \forall X \text{dog}(X) \rightarrow \text{has\_tail}(X) \)
    - now, using deduction we can conclude:
      - \( \text{has\_tail}(\text{spot}) \)
Comparison of Logics (I)

- Propositional logic:
  
  "spot is a dog" \( p \)
  
  + decidability/completeness
  - limited expressive power
  + practical deduction mechanism

  → circuit design, “answer set” programming, ...

- Predicate logic: (first order)
  
  "spot is a dog" \( \text{dog}(\text{spot}) \)
  
  +/- decidability/completeness
  +/- good expressive power
  + practical deduction mechanism (e.g., SLD-resolution)

  → classical logic programming!
Comparison of Logics (II)

- Higher-order predicate logic:
  - “There is a relationship for spot” \( X(\text{spot}) \)
    - decidability/completeness
    - good expressive power
    - practical deduction mechanism

  But interesting subsets \( \rightarrow \) HO logic programming, functional-logic programming, ...

- Other logics: decidability? Expressive power? Practical deduction mechanism?
  Often (very useful) variants of previous ones:
    - Predicate logic + constraints (in place of unification)
      \( \rightarrow \) constraint programming!
    - Propositional temporal logic, etc.

- Interesting case: \( \lambda \)-calculus
  - similar to predicate logic in results, allows higher order
    - does not support predicates (relations), only functions
  \( \rightarrow \) functional programming!
We code the problem as definite (Horn) clauses:

\[
\begin{align*}
nat(0) \\
\neg nat(X) & \lor nat(s(X)) \\
\neg nat(X) & \lor add(0, X, X) \\
\neg add(X, Y, Z) & \lor add(s(X), Y, s(Z)) \\
\neg nat(X) & \lor mult(0, X, 0) \\
\neg mult(X, Y, W) & \lor \neg add(W, Y, Z) & \lor mult(s(X), Y, Z) \\
\neg nat(X) & \lor \neg nat(Y) & \lor \neg mult(X, X, Y) & \lor \text{nat\_square}(X, Y)
\end{align*}
\]

**Query:** \( nat(s(0)) \) ?

**In order to refute:** \( \neg nat(s(0)) \)

**Resolution:**

\( \neg nat(s(0)) \) with \( \neg nat(X) \lor nat(s(X)) \) gives \( \neg nat(0) \)

\( \neg nat(0) \) with \( nat(0) \) gives \( \square \)

**Answer:** (yes)
Generating squares by SLD-Resolution – Logic Programming (II)

\[ \text{nat}(0) \]
\[ \neg \text{nat}(X) \lor \text{nat}(s(X)) \]
\[ \neg \text{nat}(X) \lor \text{add}(0, X, X) \]
\[ \neg \text{add}(X, Y, Z) \lor \text{add}(s(X), Y, s(Z)) \]
\[ \neg \text{nat}(X) \lor \text{mult}(0, X, 0) \]
\[ \neg \text{mult}(X, Y, W) \lor \neg \text{add}(W, Y, Z) \lor \text{mult}(s(X), Y, Z) \]
\[ \neg \text{nat}(X) \lor \neg \text{nat}(Y) \lor \neg \text{mult}(X, X, Y) \lor \text{nat} \_\text{square}(X, Y) \]

- **Query:** \( \exists X \exists Y \text{add}(X, Y, s(0)) \) ?

- In order to refute: \( \neg \text{add}(X, Y, s(0)) \)

- **Resolution:**
  \( \neg \text{add}(X, Y, s(0)) \) with \( \neg \text{nat}(X) \lor \text{add}(0, X, X) \) gives \( \neg \text{nat}(s(0)) \)
  \( \neg \text{nat}(s(0)) \) solved as before

- **Answer:** \( X = 0, Y = s(0) \)

- **Alternative:**
  \( \neg \text{add}(X, Y, s(0)) \) with \( \neg \text{add}(X, Y, Z) \lor \text{add}(s(X), Y, s(Z)) \) gives \( \neg \text{add}(X, Y, 0) \)
Generating Squares in a Practical Logic Programming System (I)

:- module(_,_,['bf/af']).

nat(0) <- .
nat(s(X)) <- nat(X).

le(0,_X) <- .
le(s(X),s(Y)) <- le(X,Y).

add(0,Y,Y) <- nat(Y).
add(s(X),Y,s(Z)) <- add(X,Y,Z).

mult(0,Y,0) <- nat(Y).
mult(s(X),Y,Z) <- add(W,Y,Z), mult(X,Y,W).

nat_square(X,Y) <- nat(X), nat(Y), mult(X,X,Y).

output(X) <- nat(Y), le(Y,s(s(s(s(s(0)))))), nat_square(Y,X).
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<td>?- nat(X).</td>
<td>X = 0 ; X = s(0) ; X = s(s(0)) ; …</td>
</tr>
<tr>
<td>?- add(X,Y,s(0)).</td>
<td>(X = 0 , Y=s(0)) ; (X = s(0) , Y = 0)</td>
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